A Casson-Lin invariant for knots in homology 3-spheres

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Zusammenfassung: Im Jahr 1985 definierte A. Casson eine topologische Invariante $\lambda(\Sigma)$ für Homologie-3-Sphären Σ, die, vereinfacht formuliert, nicht-abelsche SU(2)-Darstellungen der Fundamentalgruppe von Σ mit Vorzeichen zählt. X.-S. Lin griff das konstruktive Prinzip Casson's auf und definierte 1992 eine Invariante $h(k)$ für Knoten k in der 3-Sphäre S^3. Die Berechnung von $h(k)$ führt auf einen von Lin als "mysteriös" eingeschätzten Zusammenhang mit der Knotensignatur σ_k, einer klassischen Seifert-Invariante.

In der vorliegenden Arbeit werden beide Ansätze zusammengeführt, um unter Verwendung von SU(2)-Darstellungen der Knotengruppe $\pi_1(\Sigma - k)$ eine Schnittzahl $s^\alpha(k \subset \Sigma)$ zu definieren (wobei α anzeigt, dass für die Darstellung der Knotenmeridiane SU(2)-Matrizen mit Spur $2\cos\alpha$, $\alpha \in (0,\pi)$, verwendet werden). Das zentrale Resultat ist die Berechnung von $s^\alpha(k \subset \Sigma)$ mit Hilfe eines Skein-Algorithmus. Aus dieser folgt, dass $s^\alpha(k \subset \Sigma)$ eine Invariante für Knoten in Homologie-3-Sphären ist. Es ergibt sich

$$s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma) + \frac{1}{2}\sigma_{k\subset\Sigma}(e^{2i\alpha}) \ ,$$

wobei die äquivariante Signatur $\sigma_{k\subset\Sigma}(e^{2i\alpha})$ eine Verallgemeinerung der Knotensignatur darstellt. Damit bildet $s^\alpha(k \subset \Sigma)$ das topologische Gegenstück zu der von C. Herald auf analytischem Wege definierten Invariante $h_\alpha(\Sigma, k)$. Die Invarianten von Casson und Lin ergeben sich als Spezialfälle $\lim_{\alpha\to 0,\pi} s^\alpha(k \subset \Sigma)$ bzw. $s^{\pi/2}(k \subset S^3)$.

Das Resultat der Berechnung wird zeigt, dass die Bedingung $\sigma_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ hinreichend für die Existenz eines abelschen Limes nicht-abelscher Darstellungen von $\pi_1(\Sigma - k)$ ist. Insbesondere folgt, dass $\pi_1(\Sigma - k)$ in einem solchen Fall nicht-abelsche SU(2)-Darstellungen ermöglicht. Darüberhinaus werden die Zusammenhänge, die man zwischen $s^\alpha(k \subset \Sigma)$ und der klassischen Seifert-Invariante $\sigma_{k\subset\Sigma}(e^{2i\alpha})$ beobachten kann, begründet.

Abstract: In 1985 A. Casson defined a topological invariant $\lambda(\Sigma)$ for homology 3-spheres. Roughly speaking, $\lambda(\Sigma)$ counts the irreducible SU(2)-representations of the fundamental group of Σ with signs. In 1992, motivated by Casson's construction, X.-S. Lin defined an invariant $h(k)$ for knots k in the 3-sphere S^3. The computation yields a correlation to the knot signature σ_k, a classical Seifert invariant, which seemed "mysterious" to Lin.

Combining both constructions, we define an intersection number $s^\alpha(k \subset \Sigma)$ using the representations of the knot group $\pi_1(\Sigma - k)$ (where α indicates that SU(2)-matrices with trace $2\cos\alpha$, $\alpha \in (0,\pi)$, are used to represent the knot meridians). Our main result is the computation of $s^\alpha(k \subset \Sigma)$ by using a skein algorithm. The computation implies that $s^\alpha(k \subset \Sigma)$ is actually an invariant for knots in homology 3-spheres. It yields

$$s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma) + \frac{1}{2}\sigma_{k\subset\Sigma}(e^{2i\alpha}) \ ,$$

where the equivariant signature $\sigma_{k\subset\Sigma}(e^{2i\alpha})$ is a generalization of the knot signature. It turns out that $s^\alpha(k \subset \Sigma)$ is the topological counterpart of the knot invariant $h_\alpha(\Sigma, k)$ defined by C. Herald along the lines of the analytical interpretation of Casson's invariant. The invariants of Casson and Lin appear as the special cases $\lim_{\alpha\to 0,\pi} s^\alpha(k \subset \Sigma)$ and $s^{\pi/2}(k \subset S^3)$ respectively.

Using the results of the computation we show that the condition $\sigma_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ ensures the existence of an abelian limit of non-abelian representations of $\pi_1(\Sigma - k)$. In particular this implies that $\pi_1(\Sigma - k)$ admits non-abelian SU(2)-representations. Furthermore the computation of $s^\alpha(k \subset \Sigma)$ provides an explanation of the correlations between the invariant $s^\alpha(k \subset \Sigma)$ based on representation spaces and the classical Seifert invariant $\sigma_{k\subset\Sigma}(e^{2i\alpha})$.

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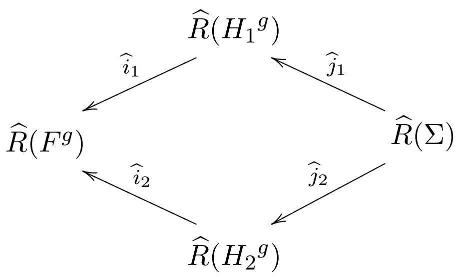
Contents

Chapter 1

Introduction

In 1985 Andrew Casson defined an invariant $\lambda(\Sigma)$ of homology 3-spheres Σ via elegant constructions on SU(2)-representation spaces. The idea of Casson's invariant comes from the observation that for a given Heegaard splitting $\Sigma = H_1^g \cup H_2^g$ (where H_1^g and H_2^g are solid handlebodies of genus g meeting along their common boundary F^g) the surjections $\pi_1(F^g) \overset{i_{*k}}{\to} \pi_1(H_k^g) \overset{j_{*k}}{\to} \pi_1(\Sigma)$, $k = 1, 2$, of the fundamental groups give rise to inclusions of spaces of conjugacy classes of non-abelian representations of these groups into SU(2):



It turns out that $\widehat{R}_k := \widehat{R}(H_k^g)$ are complementary (and equal) dimensional smooth open submanifolds of $\widehat{R}(F^g)$. Then Casson's invariant is roughly "the algebraic intersection number" of the manifolds $\widehat{R}_k$ in $\widehat{R}(F^g)$. The isotopy $\widehat{R}_1 \rightsquigarrow \widetilde{R}_1 \pitchfork \widehat{R}_2 \subset \widehat{R}(F^g)$ which is used to obtain a transversal intersection can be compactly supported away from the abelian singularities of $\widehat{R}(F^g)$. This is possible because the condition $H_1(\Sigma, \mathbb{Z}) = 0$ guarantees that the trivial representation of Σ (which is the only abelian one) is isolated in $\widehat{R}(F^g)$.

Let k be a knot in Σ and denote the manifold obtained by $\frac{1}{n}$-surgery on k by $\Sigma + \frac{1}{n}k$. Then it turns out that the difference

$$\lambda'(k) := \lambda(\Sigma + \frac{1}{n+1}k) - \lambda(\Sigma + \frac{1}{n}k) = \frac{1}{2}\Delta''_{k\subset\Sigma}(1) \tag{1.1}$$

is independent of n and therefore an invariant of the knot k. In fact $2\lambda'(k)$ is determined by the second derivative of the symmetrized Alexander polynomial $\Delta_{k\subset\Sigma}(t)$ evaluated at 1. Together with the initial value $\lambda(S^3) = 0$ equation (1.1) allows us to compute $\lambda(\Sigma)$ for any homology 3-sphere. There are many remarkable consequences due to the properties of Casson's invariant. One of the most important corollaries states that any homotopy sphere has zero Rohlin invariant. Another beautiful corollary is that a knot $k \subset S^3$ has *property P* if $\Delta''_k(1) \neq 0$. Here a non-trivial knot is said to have property P if no non trivial Dehn surgery on k yields a homotopy sphere.

In 1992 K. Walker generalized Casson's construction to the case of rational homology 3-spheres (cf. [Wak92]). Five years later C. Lescop published a global surgery formula which determines the Casson-Walker invariant as a function of the surgery data given for a link in S^3 (cf. [Les97]). Because *all* closed orientable 3-manifolds can be presented by a surgery on a link her definition can be extended to this general case.

In 1990 C. H. Taubes introduced a gauge theoretical approach to the Casson invariant by interpreting it as the Euler characteristic of the Floer homology of Σ (cf. [Tau90]). In the gauge theoretical context the equivalence classes of representations of the fundamental group are regarded as flat connections of the principal (and therefore trivial) SU(2)-bundle over Σ. As usual when using gauge theory as a tool for investigating the topology of manifolds one has to perturb the flatness conditions. These perturbations are the analytical counterpart of the topological problem to make sense of the intersection of $\widehat{R}_1$ and $\widehat{R}_2$ in $\widehat{R}(F^g)$.

Motivated by Casson's original construction, X.-S. Lin defined an invariant $h(k)$ of knots k in S^3 by counting the number of conjugacy classes of SU(2)-representations of the knot group with appropriately defined signs (cf. [Lin92]). Presenting the knot as a closed braid and representing all knot meridians by SU(2)-matrices with zero trace, Lin computed his invariant and showed

$$h(k) = \frac{1}{2}\sigma_k$$

where σ_k denotes the classical knot signature of k. In [Kro96] Lin's construction is generalized to SU(2)-representations whose values on any knot meridian have trace $2\cos\alpha$ with a specified $\alpha \in (0, \pi)$. This yields the intersection number $h^\alpha(k)$ which can be defined if the condition $\Delta_k(e^{2i\alpha}) \neq 0$ holds. Again the computation is based on a braid presentation for k and shows that $h^\alpha(k)$ is determined by the equivariant (or Tristram-Levine) signature $\sigma_k(e^{2i\alpha})$:

$$h^\alpha(k) = \frac{1}{2}\sigma_k(e^{2i\alpha}) \ , \ \ \sigma_k(\omega) := \operatorname{sign}((1-\omega)V + (1-\overline{\omega})V^T) \ , \ \omega \in S^1 \ , \tag{1.2}$$

where V denotes a Seifert matrix of k.

This is used in [HK98] to generalize a result of C. Frohman and E.P. Klassen who considered the following question: *Which are the conditions for an abelian* SU(2)*-representation* ρ_α *of a knot group (mapping the knot meridians to* SU(2)*-matrices with trace* $2\cos\alpha$*) to be a limit of non-abelian representations?*

It was shown by Klassen ([Kla91]) that the condition $\Delta_k(e^{2i\alpha}) = 0$ must be satisfied if ρ_α is to be such a limit. Moreover, it is conjectured that this condition is also sufficient.

In [FK91] the authors proved the conjecture under the assumption that $e^{2i\alpha}$ is a single root of $\Delta_k(t)$. Using relation (1.2) we showed in [HK98] that ρ_α is a limit of non-abelian representations if the equivariant signature jumps at $e^{2i\alpha}$. Therefore the conjecture holds if $e^{2i\alpha}$ is a root of $\Delta_k(t)$ of odd multiplicity. Note that these results support the conjecture that all 3-manifolds with non trivial fundamental group admit a non-trivial representation in SU(2) (see [Kir97], Problem 3.105 (A)).

The stated results were independently proven by C. Herald using the gauge theory treatment initiated by Taubes (cf. [Her97]). Since the gauge theoretical approach provides a more general viewpoint, Herald was able to show the results for knots k in *any* homology sphere Σ. His main tool is an intersection number $h_\alpha(k, \Sigma)$ defined as follows. Let m be the meridian of a knot $k \subset \Sigma$ and denote by $\mathcal{R}(T^2)$ the representation variety of the boundary torus whose fundamental group is generated by m and the canonical longitude of k. Further let S_α denote the set of representations

$\rho \in \mathcal{R}(T^2)$ such that $\operatorname{tr}\rho(m) = 2\cos\alpha$ and let $r : \mathcal{R}(\Sigma - k) \to \mathcal{R}(T^2)$ be the projection induced by the inclusion of the torus boundary of the knot complement. Then a generic perturbation h deforms $\mathcal{R}(\Sigma - k)$ into the moduli space $\mathcal{R}_h(\Sigma - k)$ so that $r(\mathcal{R}(\Sigma - k)) \cap S_\alpha$ is a transversal intersection on the pillow case $\mathcal{PC}$ (which denotes the smooth part of $\mathcal{R}(T^2)$). By counting these (finitely many) intersection points with proper signs the invariant $h_\alpha(\Sigma, k) := \langle r(\mathcal{R}_h(\Sigma - k)), S_\alpha\rangle_{\mathcal{PC}} \in \mathbb{Z}$ is defined. Herald shows that for $\alpha \in [0, \pi]$ with $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$, the formula

$$h_\alpha(\Sigma, k) = 4\lambda(\Sigma) + \frac{1}{2}\sigma_{k\subset\Sigma}(e^{2i\alpha})$$

holds. We call a formula like this, relating an intersection number defined via representation theory with topological invariants of the 3-manifold and the knot, *computational formula.*

Since the right hand side only contains topological invariants it is natural to ask: *Is it possible to define a similar Casson-Lin invariant using Casson's original topological approach?*

The intersection number $s^\alpha(k \subset \Sigma)$ defined below gives a positive answer to this question. The construction which is outlined in the following combines the techniques used to define the invariants of Casson and Lin. The construction is reflected by the result of the computation which is essentially the sum of Casson's and Lin's invariants.

Let $k \subset \Sigma$ be a knot in an arbitrary homology sphere, F a Seifert surface of k, i.e. $\partial F = k$, and $H_1^g \cup H_2^g$ be a Heegaard splitting of genus g. Taking F as a thickened 1-complex we can isotope F into H_1^g to achieve the situation $k \subset F \subset H_1^g$ as a starting point. Further let (H_1^g, h) be a Heegaard diagram associated with the Heegaard splitting of Σ where we consider a standard embedding of the handlebody H_1^g in S^3. Then the homeomorphism $h : F^g \xrightarrow{\cong} F^g$ is specified by the g curves $h(\partial D_i)$ attaching the boundaries of the handle discs D_i of the complementary handlebody H_2^g. For a knot $k \subset \Sigma$ and a Heegaard diagram with $k \subset H_1^g$ we denote the knot obtained by the standard embedding of H_1^g by $k' \subset H_1^g \subset S^3$. Thus the use of a Heegaard diagram (H_1^g, h) allows for a concrete description of the representation space of $\pi_1(\Sigma - k)$ by identifying it with the intersection

$$\widehat{R}(\Sigma - k) = \widehat{R}_1(H_1^g - k') \cap \widehat{R}_2 \subset \widehat{R}(F^g)$$

where the embedding $\widehat{h}(\widehat{R}_2)$ is still denoted by $\widehat{R}_2$.

In order to obtain a dimensional situation similar to the definition of the Casson invariant we restrict our considerations to representations which map the knot meridians to SU(2)-matrices with fixed trace $2\cos\alpha$. (All representation spaces subject to this restriction are indexed by α). Then the representation space $\widehat{R}^\alpha(H_1^g - k')$ gives rise to the $(3g-3)$-dimensional manifold $\widehat{R}'^\alpha(\beta)$ which can be properly embedded in $\widehat{R}(F^g)$ (where we assume $g \geq 2$ without loss of generality). Resembling Lin's approach we define $\widehat{R}_1'^\alpha(\beta)$ using the intersection of the $(2n+3g-6)$-dimensional *diagonal* $\widehat{\Lambda}^\alpha_{g,n}$ and the $(2n+3g-3)$-dimensional *graph* $\widehat{\Gamma}^\alpha_\beta$ which both are submanifolds of the $(4n+3g-6)$-dimensional manifold $\widehat{H}^\alpha_{g,n}$. The key tool to obtain the graph is the *standard position* for k' which is a $2n$-plat presentation $\widehat{\beta}$ whose n upper closing arcs run through exactly one handle. Here β denotes a braid with $2n$ strands. The standard position is obtained from the initial Heegaard diagram by isotopies of the knot and stabilization of the handlebodies (with the new genus still denoted by g and $g \geq n$). Then we obtain the graph from the diffeomorphism induced by the braid automorphism on the matrices representing the meridians of the closing arcs of $\widehat{\beta}$. After an isotopy $\widehat{\Gamma}^\alpha_\beta \rightsquigarrow \widetilde{\Gamma}^\alpha_\beta$ we obtain a $(3g-3)$-dimensional manifold from the transversal intersection

$$\widehat{R}_1'^\alpha(\beta) := \widetilde{\Gamma}^\alpha_\beta \pitchfork \widehat{\Lambda}^\alpha_{n,g} \subset \widehat{H}^\alpha_{n,g}\ .$$

With the g longitudes of H_1^g and the n meridians of the closing arcs as generators of $\pi_1(H_1^g - \widehat{\beta})$, the map $i_{1*} : \pi_1(F^g) \to \pi_1(H_1^g - \widehat{\beta})$ induced by inclusion is surjective. Therefore the map $\widehat{i}_1 : \widehat{\Lambda}^\alpha_{n,g} \to \widehat{R}(F^g)$ induced on the representation spaces provides a proper embedding $\widehat{R}_1'^\alpha(\beta) \hookrightarrow \widehat{R}(F^g)$ of manifolds and we may regard the plat presentation to be natural in this context.

For dimensional reasons there is an isotopy $\widehat{R}'^\alpha(\beta) \rightsquigarrow \widetilde{R}'^\alpha(\beta)$ to a transversal 0-dimensional intersection: $\widetilde{R}'^\alpha(\beta) \pitchfork \widehat{R}_2 \subset \widehat{R}(F^g)$ (where we keep the notation for the embedding of $\widehat{R}'^\alpha(\beta)$). Because the abelian representations of the knot complement are isolated if the condition $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ holds, both isotopies above can be chosen with compact support. Thus, after choosing an orientation of the manifolds, the intersection number $s^\alpha(k \subset \Sigma)$ is well defined:

Definition 1. *Let the manifolds $\widehat{R}_1'^\alpha(\beta)$, $\widehat{R}_2$ and $\widehat{R}(F^g)$ be endowed with orientations. If the condition $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ holds there exists an isotopy $\widehat{R}_1'^\alpha(\beta) \rightsquigarrow \widetilde{R}_1'^\alpha(\beta)$ with compact support such that the intersection number*

$$s^\alpha(k \subset \Sigma) := (-1)^g \sum_{p \in \widetilde{R}_1'^\alpha(\beta) \pitchfork \widehat{R}_2 \subset \widehat{R}(F^g)} \varepsilon_p \ , \ \varepsilon_p = \pm 1$$

is well defined.

The factor $(-1)^g$ is needed to ensure the invariance under stabilization.

In contrast to the proofs given in [Lin92] and [HK98] we derive the fact that $s^\alpha(k \subset \Sigma)$ is a knot invariant from a computational formula in analogy with Herald's result. As for the computation of Casson's knot invariant $\lambda'(k)$ we want to utilize the existence of a knot $k' \subset S^3$ which has the same Seifert matrix as $k \subset \Sigma$ and compute $s^\alpha(k \subset \Sigma)$ by examining a skein algorithm for k'. Here complications arise because the corresponding $k' \subset S^3$ must respect the Heegaard splitting of Σ. The latter is achieved by changing an arbitrary chosen Heegaard diagram into a *homologically flat* diagram by handle slides. The expression "homologically flat" is used because the homology classes of the g attaching curves $h(\partial D_i)$ are the same as in the case of trivial pasting of H_2^g. Then a trivial embedding of H_1^g in S^3 yields k'. Note that the handle slides generally change the embedding of $k' \subset H_1^g \subset S^3$. The construction has immediate corollaries:

Corollary 2. *(a) In each free homotopy class γ of loops in a homology 3-sphere Σ all Laurent polynomials $p(t)$ with $p(t) = p(t^{-1})$ and $p(1) = 1$ are realized as Alexander polynomials of knots $k \subset \Sigma$.*
(b) Let $k_0 \subset \Sigma$ be a knot with trivial Alexander polynomial. Then all Alexander polynomials are realized by k_0 with one crossing changed.

Since the stabilizations which are applied to obtain the standard position for k' leave a homologically flat Heegaard diagram flat we can assume a homologically flat Heegaard diagram given by (H_1^g, h) with $k' \subset H_1^g \subset S^3$ in standard position.

The starting point for the computation of $s^\alpha(k \subset \Sigma)$ is the following observation. Because $s^\alpha(k \subset \Sigma)$ cannot change at α if the condition $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ holds, $s^\alpha(k \subset \Sigma)$ is constant (with respect to α) if k is a knot with trivial Alexander polynomial. From the definition of $s^\alpha(k \subset \Sigma)$ it follows that $\lim_{\alpha\to 0,\pi} s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma)$ (where the factor 2 appears because *all* intersection points are counted). Since $s^\alpha(k \subset \Sigma)$ is locally constant, the equation $s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma)$ holds for *all* α if k is a knot with $\Delta_{k\subset\Sigma} = 1$. Thus to compute $s^\alpha(k \subset \Sigma)$ we may consider an "unknotting process" which leads to a knot k_0 with trivial Alexander polynomial and observe the difference $\Delta s^\alpha(k \subset \Sigma) := s^\alpha(k_+ \subset \Sigma) - s^\alpha(k \subset \Sigma)$ where k_+ denotes the knot k with one crossing changed.

To switch a certain crossing we establish a *computational position* $k'_c = k'_c(k', k'_+) = \widehat{\beta}_c$. This position is obtained by a stabilization of the plat $\widehat{\beta}$ presenting k' followed by a stabilization of the standard splitting such that the crossing is isolated in an additional handle. Then we perform a 1-surgery along the meridian of the new handle. Note that these manipulations are in general not realizable by band homotopies of the Seifert surface $F \subset H_1^g$ bounding k. But for our purposes it is sufficient that the Alexander polynomials of $k_+ \subset \Sigma$ and $k'_+ \subset S^3$ coincide.

To control $\Delta s^\alpha(k \subset \Sigma)$ we project the (in general high dimensional) manifolds $\widetilde{R}_1'^\alpha(\beta_c)$ and $\widehat{R}(H_2^{g+1})$ via $\widehat{p} : \widehat{R}(F^{g+1}) \to \mathcal{PC}$ onto the pillow case representing the non-central part of the longitude l_0 and the meridian m_0 of the new handle. Note that due to the conditions of the projection the representations $\rho(l_0)$ and $\rho(m_0)$ commute. Then we obtain $\Delta s^\alpha(k \subset \Sigma)$ by comparing the intersection number of the projection curve of the knot $\widehat{p}^\alpha(\beta_c) := \widehat{p}(\widetilde{R}_1'^\alpha(\beta_c))$ and the projection curve of $\widehat{R}(H_2^{g+1})$, denoted by $\widehat{h}_2$, with the intersection number $\langle \widehat{p}^\alpha(\beta_c), \widehat{h}_+ \rangle_{\mathcal{PC}}$ (where $\widehat{h}_+$ denotes the projection curve of $\widehat{R}(H_2^{g+1})$ after the 1-surgery). Because there is no isotopy needed in a sufficiently small neighborhood of the abelian representations of $\widehat{R}_1'^\alpha(\beta_c)$ (and $\widehat{R}(H_1^g - k')$ resp.), the endpoints of $\widehat{p}^\alpha(\beta_c)$ on the pillow case are independent from the isotopies chosen to define $s^\alpha(k \subset \Sigma)$. Thus $\Delta s^\alpha(k \subset \Sigma)$ is determined by the endpoints of the projection curve of the knot which are given by the limit

$$\lim \widehat{p}^\alpha(\beta_c) = (\frac{1}{2} \arg \lambda_0^\alpha(k, k_+), -2\alpha) \cup (\pi - \frac{1}{2} \arg \lambda_0^\alpha(k, k_+), 2\alpha) \subset \mathcal{PC} ,$$

where

$$\lambda_0^\alpha(k, k_+) = \frac{\Delta_{k_+ \subset \Sigma}(t) - t \Delta_{k \subset \Sigma}(t)}{\Delta_{k_+ \subset \Sigma}(t) - t^{-1} \Delta_{k \subset \Sigma}(t)} \in S^1 \quad \text{and} \quad t = e^{2i\alpha} .$$

Since the skein relation for $s^\alpha(k \subset \Sigma)$ fits together with the skein algorithm for the equivariant signature, we obtain our main result.

Theorem 3. *Let $k \subset \Sigma$ be a knot and $\alpha \in (0, \pi)$ with $\Delta_{k \subset \Sigma}(e^{2i\alpha}) \neq 0$ be given. Then*

$$s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma) + \frac{1}{2} \sigma_{k \subset \Sigma}(e^{2i\alpha}) .$$

The result depends on the orientations of Σ and of the knot by an overall sign of $\lambda(\Sigma)$ and $\sigma_{k \subset \Sigma}$ respectively. Since all computations are independent from the chosen Heegaard splitting we have proven:

Theorem 4. *Suppose that the assumptions of theorem 3 hold. Then $s^\alpha(k \subset \Sigma)$ defines an invariant for knots in homology 3-spheres.*

Using a bordism argument as in [HK98] the following theorem is immediate.

Theorem 5. *Let $k \subset \Sigma$ be a knot in a homology 3-sphere and $\alpha \in [0, \pi]$ such that $\Delta_{k \subset \Sigma}(e^{2i\alpha}) = 0$.*
(a) Then the abelian representation ρ_α is a limit of non-abelian representations of $\pi_1(\Sigma - k)$ if $\sigma_{k \subset \Sigma}$ jumps at $e^{2i\alpha}$.
(b) If $\sigma_{k \subset \Sigma}(e^{2i\alpha})$ changes its value on the unit circle then the representation space of $\Sigma - k$ is non-trivial.

Remark 6.
1. The equivariant signature changes its value if $e^{2i\alpha}$ is a root of $\Delta_{k \subset \Sigma}(e^{2i\alpha})$ of odd multiplicity.

2. The invariant $h^\alpha(k)$ defined in [Kro96] and the results of [HK98] are included as the case $\Sigma = S^3$. Of course we obtain Lin's invariant for $\Sigma = S^3$ and $\alpha = \pi/2$.

We conclude the introduction with some remarks on the results and ideas regarding possible developments based on the construction.

Since our invariants of Casson type can be computed from a corresponding knot k' in S^3, it is clear that they cannot carry more topological information than the Seifert surface which bounds k and k'. Therefore it is not surprising that the invariants computed from constructions on SU(2)-representation spaces are determined by Seifert invariants like the derivative of the Alexander polynomial or the equivariant signature. It is because of this limitation that G. Burde's proof of property P for 2-bridge knots (see [Bur90]) cannot be generalized with the help of Casson type invariants. Because there is no need to isotope the representation spaces in the case of the 2-bridge knots Burde can make use of the full potential of the SU(2)-representations to detect even such non-abelian phenomena which fail to be noticed by the Casson invariant.

For $s^\alpha(k \subset \Sigma)$ this potential is lost even twice. Firstly as a consequence of the perturbation $\widehat{\Gamma}^\alpha_\beta \rightsquigarrow \widetilde{\Gamma}^\alpha_\beta$ implicating the determination of the intersection number by the local structure of the representation space at the *abelian* representations. The second loss comes with the perturbation $\widehat{R}'^\alpha_1(\beta) \rightsquigarrow \widetilde{R}'^\alpha_1(\beta)$ which puts us in the position to compute $s^\alpha(k \subset \Sigma)$ from a knot k' in S^3. Considering Herald's picture on the pillow case representing the torus boundary of $\Sigma - k$ we see that Casson type invariants in spite of their limitations may provide non trivial contributions for proving property P. Hence a more detailed comparison of the topological and the gauge theoretical approach to this invariants seems to be of further interest.

The projection curve of the knot in our construction provides explicit information about the neighborhood of an abelian representation. From this we expect some progress in proving the conjecture that $\Delta_{k\subset\Sigma}(e^{2i\alpha}) = 0$ is a sufficient condition for ρ_α to be a limit of non-abelian representations.

Another topic for further examination is the definition of a Casson-Lin invariant for knots in rational homology 3-spheres. For this, a Heegaard diagram whose gluing curves are of trivial *rational* homology seems to be a good starting point for computing a skein algorithm. For the knot invariant in the rational case we conjecture an analogous formula where the Casson invariant take values in $\mathbb{Q}$.

The paper is organized as follows. In Chapter 2 the basic facts and notations are presented. Furthermore it includes the construction of a homologically flat Heegaard diagram which is the central tool for the computation of $s^\alpha(k \subset \Sigma)$ in terms of the Casson invariant and signature function. The invariant itself is defined in chapter 3. In Chapter 4 the computation of $s^\alpha(k \subset \Sigma)$ is discussed. The example of the $(2, n)$-torus knots provides a good insight into the computational procedure and is presented in detail in section 4.1. The difference cycle which plays a central role in the computation of $s^\alpha(k \subset \Sigma)$ and $\lambda(\Sigma)$ is the subject of the comparison of both invariants in section 4.2. Section 4.3 contains the computation of $s^\alpha(k \subset \Sigma)$ for arbitrary knots and presents our main results. These are applied in section 4.4 to obtain the statements on the representation space of the knot complement $\Sigma - k$.

Chapter 2

Facts and Notations

2.1 Homology 3-spheres and surgery

A homology 3-sphere Σ is defined to be a 3-dimensional closed oriented manifold with the integer homology of the 3-sphere, i.e.

$$H_i(\Sigma, \mathbb{Z}) = H_i(S^3, \mathbb{Z}) = \begin{cases} \mathbb{Z}\,, & i = 0, 3 \\ 0\,, & \text{otherwise} \end{cases} \quad .$$

Being a 3-dimensional manifold, Σ admits a Heegaard splitting which is a decomposition into two handlebodies H_i^g, $i = 1, 2$, of genus g:

$$\Sigma = H_1^g \cup H_2^g \ .$$

The handlebodies meet along their common boundaries F^g where F^g denotes the closed oriented surface of genus g ([Hem76], Th.2.5).

A convenient way of viewing and constructing a 3-manifold M is provided by a *Heegaard diagram* associated with a given Heegaard splitting (see [Hem76], p.17f).

Definition 2.1.1. *Let a Heegaard splitting $M = H_1^g \cup H_2^g$ be given. Further let the handlebody H_1^g be trivially embedded in S^3. Then M is determined by the homeomorphism $h : F^g \to F^g$ which glues in the complementary handlebody H_2^g. The homeomorphism h itself is determined by the images of the boundaries of the g meridian discs $D_i \subset H_2^g$ (which are the g non intersecting, 2-sided "gluing curves" on F^g). Then the collection $(H_1^g, H_2^g, h(\partial D_1), \ldots, h(\partial D_g)) =: (H_1^g, h)$ is called a Heegaard diagram of M. (Sometimes a Heegaard diagram of M is denoted by $H_1^g \cup_h H_2^g$.)*

Because any closed oriented 3-manifold can be triangulated (cf. [MKS66], [New29]), we obtain a Heegaard splitting from a chosen triangulation. The construction is as follows. One handlebody is the result of thickening up the 1-skeleton. Its handles are built from the edges and connect the 3-balls which for their part are the result of the thickened points. The complementary handlebody consists of the thickened plates as handles which connect the 3-balls inside the tetrahedron (compare the proof of Th.2.5 in [Hem76]). It is clear, for example because of the dependency of the triangulation, that a Heegaard splitting cannot be unique. Moreover, for a chosen decomposition with genus g we obtain another decomposition with genus $g + 1$ by adding an unknotted handle to each of the handlebodies. Here "unknotted" means that it is possible to span a disc in the new handle (see [Sav99], Ch.1.3) which furthermore ensures that the complement remains a handlebody. The process described is called *stabilization* and two decompositions are said to be *stably* *equivalent* if there exists a sequence of stabilizations which takes one decomposition into the other. The following result is due to Singer ([Sin33]).

Theorem 2.1.2. *Any two Heegaard-decompositions of a closed oriented 3-manifold are stably equivalent.*

Another way of describing a closed oriented 3-manifold is given by surgery on a framed link $L \subset S^3$. The fundamental surgery process is to cut out an open tubular neighborhood of one link component and to past a solid torus back as it is given by the framing coefficients (compare [Kir78] for the case of integer framing and [Rol84] for the more general case of rational coefficients). The importance of the surgery concept might be underlined by the following result of Lickorish and Wallace ([Lic62], [Wal60]).

Theorem 2.1.3. *Any closed oriented 3-manifold can be obtained by surgery on a framed link L in* S^3.

Remark 2.1.4.
1. We denote a 3-manifold M given by a surgery on a framed link $L = l_1 \cup \ldots \cup l_n$ by $M = c_1 l_1 + \ldots + c_n l_n$ where the c_i denote the framing coefficients.
2. It can be shown that it is possible to obtain any manifold M by a ± 1-surgery on a link in S^3, i.e. $M = c_1 l_1 + \ldots c_n l_n$ with $c_i = \pm 1$.(see [FM97], Th.9.5).

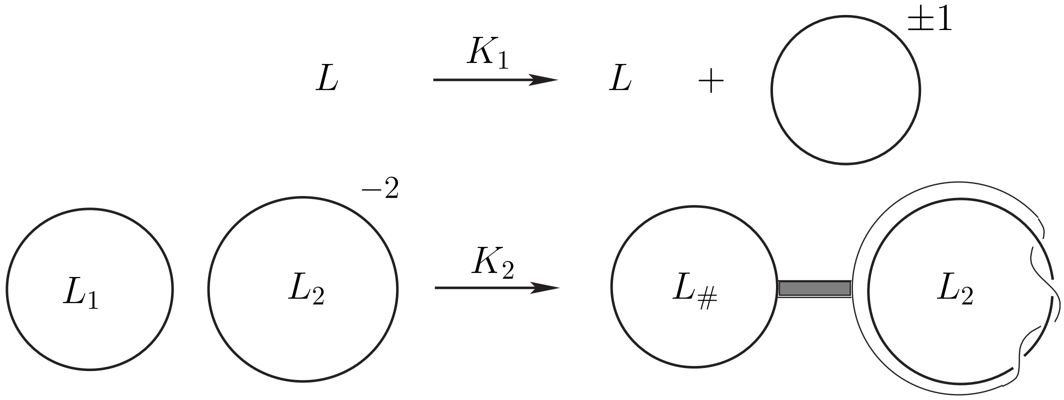


Figure 2.1: The Kirby moves K_1 and K_2.

It is immediate that two 3-manifolds are homeomorphic if they differ by a Kirby move K_1 or K_2. Kirby proved that the reversed direction holds as well ([Kir78]):

Theorem 2.1.5. *Let M and M' be closed oriented 3-manifolds which are given by surgery on the framed links L and L' respectively. Then M is homeomorphic to M' if and only if there exists a sequence of Kirby moves K_1 and K_2 which takes L into L' (as framed links).*

2.2 Seifert-invariants of knots in homology 3-spheres

An important class of invariants for a knot k in the 3-sphere is based on the existence of a *Seifert surface* F, where F is an oriented surface which bounds the knot. Because the first and second homology of a homology 3-sphere Σ vanish, a Seifert surface exists even for a knot in Σ. In the more general case of a *link* in a homology sphere we have ([Sav99], Th.7.12):

Theorem 2.2.1. *Let $L \subset \Sigma$ be a link in a homology 3-sphere Σ. Then there exists an oriented surface $F \subset \Sigma$ with $\partial F = L$. F is called a Seifert surface for L.*

Therefore we are able to generalize all knot invariants constructed with the help of a Seifert surface to the case of knots in arbitrary homology-3-spheres. We call invariants of that kind *Seifert invariants* and denote them by $S(k \subset \Sigma)$. We will make frequent use of the following Seifert invariants: the *Alexander polynomial* $\Delta_{k\subset\Sigma}(t)$ and the *equivariant signature* $\sigma_{k\subset\Sigma}(\omega)$. The definitions are sketched below.

Let $k \subset \Sigma$ be a knot in a homology 3-sphere and $F \subset \Sigma$ a Seifert surface for k. Since F is oriented a normal direction is given. Thus we can push cycles on F along this normal direction into the complement of F. This defines a homomorphism $H_1(F) \to H_1(\Sigma - F)$ with $x \mapsto x^+$ where $H_1(X) := H_1(X, \mathbb{Z})$. The Seifert pairing $H_1(F) \otimes H_1(F) \to \mathbb{Z}$ is given by $x \otimes y \mapsto \text{ lk } (x, y^+)$ where lk is the linking number in Σ (see [Sav99], Ch.7.5). By fixing a base $\{a_i | 1 \le i \le 2g\}$ of $H_1(F)$ the pairing is described by a $2g \times 2g$-matrix V over $\mathbb{Z}$. Here g denotes the genus of F and V is called a *Seifert matrix* for k. The antisymmetric matrix $V - V^T$ (V^T being the transpose of V) is the intersection matrix of the basis $\{a_i\}$ in $H_1(F)$ ([BZ85], Ch.8b).

Definition 2.2.2. *Let $k \subset \Sigma$ be a knot in a homology-3-sphere and V_s a Seifert matrix of k. Then*

$$\Delta_{k\subset\Sigma}(t) = \text{ det } (t^{\frac{1}{2}} V - t^{-\frac{1}{2}} V^T)$$

is the normalized Alexander polynomial of k where normalized means $\Delta_{k\subset\Sigma}(t) = \Delta_{k\subset\Sigma}(t^{-1})$ and $\Delta_{k\subset\Sigma}(1) = 1$.
Under the same assumptions as above let $\omega \in S^1$. Consider the hermitian matrix

$$H(\omega) = (1 - \omega)V + (1 - \bar{\omega})V^T = (\omega^{-\frac{1}{2}} - \omega^{\frac{1}{2}})(\omega^{\frac{1}{2}} V - \omega^{-\frac{1}{2}} V^T) \; . \tag{2.1}$$

Then the equivariant signature (or Tristram-Levine-signature) $\sigma_{k\subset\Sigma}(\omega)$ is defined to be the signature of $H(\omega)$, i.e.

$$\sigma_{k\subset\Sigma}(\omega) = \text{sign}(H(\omega)) \; .$$

The signature function $\sigma_{k\subset\Sigma} : S^1 \to \mathbb{Z}$ is the map $\omega \mapsto \sigma_{k\subset\Sigma}(\omega)$ for $\omega \neq 1$ and $\sigma_{k\subset\Sigma}(1) = 0$. Let $z_k = \{\omega \in S^1 | \Delta_{k\subset\Sigma}(\omega) = 0\}$ be the zeros of $\Delta_{k\subset\Sigma}$ on the unit circle. Then it follows from equation (2.1) that $\sigma_{k\subset\Sigma}(\omega)$ is constant on the components of $S^1 - z_k$. Moreover, it can be seen that $\sigma_{k\subset\Sigma}(\omega) = 0$ if ω lies in a small neighborhood of 1 (for details see [Kau87], Ch.12 and [Gor78]). If the definition of a knot invariant is founded on the Seifert matrix V there are strong connections between the cases of S^3 and arbitrary homology 3-spheres.

Let k be a knot in the homology 3-sphere Σ which is given by a surgery on a framed link $L \subset S^3$. Note that the invariant $S(k \subset \Sigma)$ does not change if Σ is manipulated by an additional surgery on a new link component $l \subset S^3$ and if the latter is boundary to k ([Sav99], Lem.7.14). Here "being boundary" means that k and l possess disjoint Seifert surfaces. Let a ± 1-surgery description of Σ be given (see Rem.2.1.4). Isotope the Seifert surfaces of the n-th component and that of k into discs with thin bands attached. Then local homotopies lead to a situation where l_n and k are boundary. It should be remarked that the homotopy may change the link but it preserves the isotopy classes of both, the knot k and the link component l_n. An inductive procedure, considering k as a knot in the homology 3-sphere $\Sigma' = c_1 l_1 + \ldots + c_{n-1} l_{n-1}$ in the second step, shows

Theorem 2.2.3. *Let $k \subset \Sigma$ be a knot in a homology 3-sphere and $S(k \subset \Sigma)$ a Seifert invariant. Then there exists a knot $k' \subset S^3$ with $S(k' \subset S^3) = S(k \subset \Sigma)$.*

By using a special Heegaard splitting as a starting point and pushing the knot entirely into H_1^g we receive a more vivid version of theorem 2.2.3. The construction will provide a very useful point of view because then the knot $k' \subset S^3$ given in 2.2.3 is in fact the knot $k' \subset H_1^g$ resulting from a trivial embedding $H_1^g \subset S^3$. To obtain the Heegaard splitting with the property which we call *homologically flat* we perform handle slides with respect to the meridian discs of the glued in H_2^g and embedded discs of H_1^g.

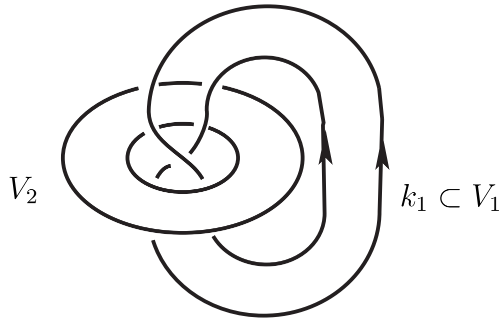


Figure 2.2: The case $k_1 = \widehat{\sigma_1} \subset V_1 \subset S^3$.

Example 2.2.4. Let us approach the problem by discussing one of the most simple cases: the $(2,1)$-torus knot presented by a closed braid, i.e. $k_1 := \widehat{\sigma_1} \sim k_0$, which is embedded in the full torus $H_1^1 =: V_1$ (see Fig.2.2). Performing a +1-surgery along the core of the complementary full torus $H_2^1 =: V_2$ yields the trefoil $k_3 = \widehat{\sigma_1^3} \subset S^3$. Thus the embedding $k_1 \subset V_1 \subset S^3$ does not carry the full "knotting information" of k_3 in S^3. Moreover, all linking phenomena computed from V_1 as being embedded in S^3 fail to notice the additional linking provided by the exterior twist. The twist is a consequence of the meridian part of the gluing curve: $h_*(\partial D) = l+m \in \pi_1(\partial V_1) = \pi_1(T^2) = H_1(T^2)$ where ∂D denotes the boundary of the meridian disc in H_1^2.

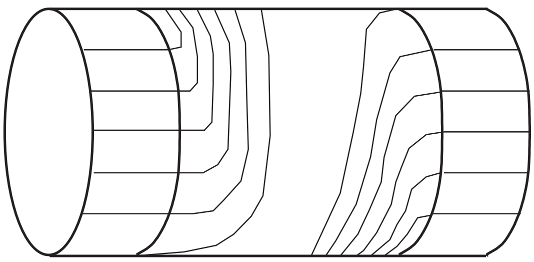


Figure 2.3: A simple Dehn twist along the meridian of a handle.

To obtain the complete knotting information already in V_1 a Dehn twist along the meridian disc of V_1 is needed. Being a handle slide this manipulation does not change the 3-manifold which remains S^3 in our example. Now the gluing curve is represented by $l \in \pi_1(T^2)$. We call curves of this kind, carrying no homological meridian twists, *homologically flat*. The proof of the next theorem will show that establishing a Heegaard diagram with gluing curves being homologically flat is sufficient for "seeing" the required knotting (or linking) information already in $H_1^g \subset S^3$.

Remark 2.2.5. In 1989 C.M. Gordon and J. Luecke proved the famous result that knots in S^3 are determined by their complements ([GL89]). This is not the case for links in S^3. Regarding the core of the complementary torus V_2 together with k_1 and k_3 respectively as links in S^3 the example above provides a well-known counterexample (cf. [Rol76], Ch.3.A.2).

Due to its importance we shall give a precise definition of a homologically flat Heegaard diagram.

Definition 2.2.6. *Let a Heegaard splitting of a homology 3-sphere Σ be given. Furthermore choose the longitudes l_i and the boundaries of the meridian discs m_i of H_1^g, $1 \leq i \leq g$, as generators of $H_1(F^g)$. Then a Heegaard diagram (H_1^g, h) of Σ is homologically flat if the following condition holds: the gluing curves have the homology $h_\#(\partial D_i) = l_i \in H_1(F^g)$ where $\#$ indicates the map induced in homology.*

The possibility to establish a homologically flat Heegaard diagram reflects the fact that Σ is a homology sphere. The homologically flat Heegaard diagram can be regarded as a homologically trivial pasting of H_2^g.

Theorem 2.2.7. *Each homology 3-sphere admits a homologically flat Heegaard diagram.*

Proof. Let (H_1^g, h) be a Heegaard diagram associated with the given Heegaard splitting of Σ (compare Def.2.1.1) and choose the longitudes l_i, $1 \leq i \leq g$, of H_1^g as generators of the first homology. Furthermore choose the longitudes and the boundaries of the meridian discs of H_1^g as generators of $H_1(F^g)$, i.e.:

$$H_1(F^g) = \langle m_i, l_i, 1 \leq i \leq g | - \rangle \, .$$

If i denotes the inclusion $i : F^g \to H_1^g$ we obtain $i_\# \circ h_\#(\partial D_i) = \sum_j \lambda_{ij} l_j$, $\lambda_{ij} \in \mathbb{Z}$, and thus a presentation of $H_1(\Sigma)$ by

$$H_1(\Sigma) = \langle l_1, \ldots, l_g \mid \sum_{j=1}^{g} \lambda_{ij} l_j = 0, 1 \leq i \leq g \rangle \, .$$

Because Σ is an integer homology 3-sphere the matrix $(\lambda_{ij}) \in \mathrm{GL}(g, \mathbb{Z})$ is invertible over $\mathbb{Z}$. Let $(\mu_{ij}) \in \mathrm{GL}(g, \mathbb{Z})$ denote its inverse:

$$\delta_{ij} = \sum_k \mu_{ik} \lambda_{kj} = \sum_l \lambda_{il} \mu_{lj} \, .$$

Then handle slides on the meridian discs of H_2^g induce row operations of (λ_{ij}). Thus suitable handle slides on the meridian discs of H_2^g lead to gluing curves $h'(\partial D_i)$ which are homologically diagonal with respect to the l_i, i.e.:

$$h'_\#(\partial D_i) = \sum_j \alpha_{ij} m_j + \sum_j \delta_{ij} l_j = \sum_j \alpha_{ij} m_j + l_i \, .$$

Regarding the union of curves $h'(\partial D_i)$ as a link in S^3 we interpret the non diagonal elements of (α_{ij}) as the linking coefficients of this link. This implies $\alpha_{ij} = \alpha_{ji}$. To proceed let us orient the generators of $\pi_1(F^g)$ by a symplectic basis of $H^1(F^g, \mathbb{R})$ (cf. [Sav99], Cor.16.6). Then the twists along the discs $d_{ij} \subset H_1^g$, which are spanned between the i-th and j-th handle, $i \neq j$, (see Fig.2.4), add $\pm(m_i - m_j)$ and $\pm(m_j - m_i)$ in the homology of the curves $h'(\partial D_i)$ and $h'(\partial D_j)$ respectively. Let $\partial d_{ij} \rightsquigarrow \widetilde{\partial d_{ij}}$ be an isotopy which yields a transversal intersection $\widetilde{\partial d_{ij}} \pitchfork h'(\partial D_m) \subset F^g$ for

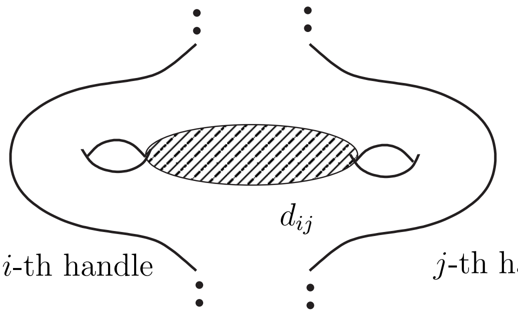


Figure 2.4: The disc d_{ij} spanned in between the i-th and j-th handle.

$m \neq i,j$. Because of $\lambda_{ij} = \delta_{ij}$ follows $\sum \varepsilon_p = 0$, $p \in \widetilde{\partial d_{ij}} \pitchfork h'(\partial D_m) \subset F^g$, $m \neq i,j$. Thus $-\alpha_{ij}$-twists along d_{ij} do not change the α_{mn}, $\{m,n\} \neq \{i,j\}$, and we can diagonalize (α_{ij}). Note that any operation adds $\mp\alpha_{ij} = \mp\alpha_{ji}$ to the diagonal elements α_{ii} and α_{jj}. Finally Dehn twists along the handle meridians of H_1^g lead to the identity matrix $\alpha_{ij} = \delta_{ij}$. Since the twists along the discs inside H_1^g are handle slides of H_1^g they do not alter Σ and the proof is complete. ■

As a starting point for the next theorem we establish the following situation: For a given homologically flat Heegaard diagram of Σ we choose a Seifert surface $F \subset \Sigma$ of k and push F inside H_1^g. This is possible since both H_2^g and F_s are thickened 1-complexes. Of course we may also start with an arbitrary Heegaard diagram. Then the twists along the discs $d_{ij} \subset H_1^g$ which are applied to obtain the flat diagram will in general change the embedding $k' \subset F \subset H_1^g \subset S^3$.

Theorem 2.2.8. *Let $k \subset \Sigma$ be a knot in a homology 3-sphere and F a Seifert surface of k with $F \subset H_1^g$ for a given Heegaard splitting of Σ. Further let (H_1^g, h) be a homologically flat Heegaard diagram associated with the splitting of Σ. Then $S(k \subset \Sigma) = S(k' \subset S^3)$ where k' denotes the knot $k' \subset H_1^g \subset S^3$.*

Proof. To show that k' is a suitable knot we have to check that the linking numbers computed with the help of the Seifert surface spanning $k' \subset S^3$ and $k \subset \Sigma$ are equal. Hence we start with some general observations for linking numbers in homology 3-spheres.
Let γ_1 and γ_2 be two embedded non intersecting curves in H_1^g. Because the linking numbers are symmetric, $\mathrm{lk}(\gamma_1, \gamma_2) = \mathrm{lk}(\gamma_2, \gamma_1)$ (cf. [Sav99], Ch.3.1), we may compute $\mathrm{lk}(\gamma_1, \gamma_2)$ with respect to the complement of γ_1. It can be seen by a Mayer-Vietoris-sequence that $H_1(H_1^g - \gamma_1)$ is free generated by the longitudes of H_1^g and the meridian m_{γ_1} of γ_1, i.e. $H_1(H_1^g - \gamma_1) = \langle l_1, \ldots, l_g, m_{\gamma_1} | - \rangle$. Gluing in the handlebody H_2^g, the images of the boundaries ∂D_i of its meridian discs in $H_1(H_1^g - \gamma_1)$ are:

$$h_\#(\partial D_i) = \sum_j \lambda_{ij} l_j + \alpha_{i\gamma_1} m_{\gamma_1}, \ \alpha_{i\gamma_1} \in \mathbb{Z} \ . \tag{2.2}$$

Thus a presentation of $H_1(\Sigma - \gamma_1)$ is given by

$$\begin{aligned} H_1(\Sigma - \gamma_1) &= \langle l_1, \ldots, l_g, m_{\gamma_1} \mid \sum_j \lambda_{ij} l_j + \alpha_i m_{\gamma_1} = 0 \ , \ 1 \leq i \leq g \rangle \\ &= \langle l_1, \ldots, l_g, m_{\gamma_1} \mid l_i = -\sum_j \mu_{ij} \alpha_j m_{\gamma_1} \ , \ 1 \leq i \leq g \rangle = \langle m_{\gamma_1} | - \rangle \ , \end{aligned} \tag{2.3}$$

where the second equation only holds for an integer homology 3-sphere, i.e. $\mu_{ij} \in \mathbb{Z}$. (Of course the result follows immediately (but less concrete) from a Mayer-Vietoris-sequence for the complement of γ_1.)
To compute the linking number we use the definition which relates the homology class of γ_2 to the generator m_{γ_1} of $H_1(\Sigma - \gamma_1)$, i.e. (cf. [Sav99], Ch.3.1 (3)):

$$H_1(\Sigma - \gamma_1) \ni [\gamma_2] = \mathrm{lk}(\gamma_1, \gamma_2)\, m_{\gamma_1} \ .$$

In $H_1(H_1^g - \gamma_1)$ we have $[\gamma_2] = \sum_j \beta_j l_j + \mu m_{\gamma_1}$. Using the relations in (2.3) we substitute the l_j and obtain

$$[\gamma_2] = (\mu - \sum_{i,j} \beta_j \mu_{ji} \alpha_i) m_{\gamma_1} \ .$$

Then for the linking number follows

$$\mathrm{lk}(\gamma_1, \gamma_2) = \mu - \sum_{i,j} \beta_j \mu_{ji} \alpha_i \ . \qquad (2.4)$$

Considering H_1^g as embedded in the 3-sphere, μ can be regarded as the "visible" linking number. The additional part in (2.4), $-\sum_{i,j} \beta_j \mu_{ji} \alpha_i$, vanishes for a homologically flat embedding of H_1^g. To see this we consider the map $i_\# : H_1(F^g) \to H_1(H_1^g - \gamma_1)$ induced by inclusion with $l_i \mapsto l_i$ and $m_i \mapsto k_i m_{\gamma_1}$. The $k_i \in \mathbb{Z}$ are given by $[\gamma_1] = k_i l_i \in H_1(H_1^g - \gamma_1)$ and count how often γ_1 runs *oriented* around the i-th handle. For the homology of the gluing curves we obtain

$$i_\# \circ h_\#(\partial D_i) = \sum_j \alpha_{ij} k_j m_{\gamma_1} + \sum_j \lambda_{ij} l_j = \alpha_{i\gamma_1} m_{\gamma_1} + \sum_j \lambda_{ij} l_j \ .$$

Therefore the flat situation with $\alpha_{ij} = 0$ implies $\alpha_{i\gamma_1} = \sum_j \alpha_{ij} k_j = 0$, $1 \le i \le g$, and we obtain

$$\mathrm{lk}(\gamma_1, \gamma_2) = \mu$$

from equation (2.4). Because the generators of $H_1(F)$ with $k = \partial F \subset \Sigma$ and $k' = \partial F \subset S^3$ can be represented by curves embedded in H_1^g, for the Seifert matrices follows $V(k \subset \Sigma) = V(k' \subset S^3)$. Then the statement is an immediate consequence. ■

Remark 2.2.9.
1. For rational homology 3-spheres the matrix (λ_{ij}) is not invertible over the integers. (Note that this is even true for the lens spaces and should not be mixed up with the statement that the mapping class group of the torus T^2 is given by $\mathrm{SL}(2, \mathbb{Z})$.) Thus the construction of $k' \subset S^3$ given in theorem 2.2.8 is not applicable in the rational case.
2. Theorem 2.2.8 fits together with the definition of linking numbers for links in handlebodies given by U. Kaiser (cf. A.3.2 and example A.3.4 in [Kai96]).

2.3 Representation spaces

As mentioned before any closed oriented 3-manifold can be triangulated. Therefore we are concerned with finitely generated fundamental groups and their representation spaces.
Thus let G be a finitely generated group. The space of all representations of G in the Lie group $\mathrm{SU}(2)$ is denoted by $R(G) = \mathrm{Hom}\ (G, \mathrm{SU}(2))$. Note that $R(G)$ is a topological space via the

compact open topology where G carries the discrete and SU(2) the usual topology which is induced from the identification $\mathrm{SU}(2) \cong S^3$ ([Sav99], Ch.14.1). A representation $\rho \in R(G)$ is called abelian (or central or trivial) if and only if its image is an abelian (or central or trivial resp.) subgroup of SU(2). Note that for SU(2) a representation ρ is abelian if and only if it is reducible (cf. [Sav99], Ch.14.2). The set of abelian (central) representations of G is denoted by $S(G)$ ($C(G)$).
Two representations ρ and ρ' are said to be conjugate ($\rho \sim \rho'$) if and only if they differ by an inner automorphism of SU(2). If, on the other hand, we identify SU(2) with S^3, it can be seen that $\mathrm{SO}(3) = \mathrm{SU}(2)/\pm 1$ acts freely on SU(2) via conjugation. Identifying SU(2) with the unit quaternions $\mathbb{H}_1$ we have an explicit interpretation of the action in terms of rotating the spatial part of the quaternion which is conjugated (see Th.2.5).

Definition 2.3.1. *Let $R(G) - S(G) =: R^{irr}(G)$ be the set of non-abelian (i.e. irreducible)* SU(2)*-representations of the finitely generated group G. Then the spaces of conjugation classes of all and of non abelian representations from G into* SU(2) *are denoted by $\mathcal{R}(G)$ and $\widehat{R}(G)$ respectively, i.e.*

$$\mathcal{R}(G) = R(G)/\mathrm{SO}(3) \quad , \quad \widehat{R}(G) = R^{irr}(G)/\mathrm{SO}(3) \; .$$

Generally $\widehat{R}(G)$ is called the *representation space of G.*

Remark 2.3.2. If there are only abelian representations of G it is quite handy to use the same notation for the equivalence classes of the non-central representations $R^{nc}(G) := R(G)-C(G)$, i.e.: $\widehat{R}(G) = R^{nc}(G)/\mathrm{SO}(3)$.

Regarding SU(2) as $S^3 \subset \mathbb{R}^4$, the space $R(G)$ has the structure of a real algebraic set, i.e. it is given by a set of polynomial equations in $\mathbb{R}^{4n}$. Being an image of a polynomial map the spaces $\mathcal{R}(G)$ and $\widehat{R}(G)$ have the structures of semi-algebraic sets. Here a set is called semi-algebraic if it is given by a finite union of finite intersections of sets defined by polynomial equations (see [Heu98] and [Sav99], Ch.14.1, for details). The following representation spaces are well known.

2.3.1 The representation space of $\pi_1(H^g)$

The fundamental group of a handlebody H^g of genus g is a free group with g generators which can be identified with the g longitudes of H^g. Therefore $R(H^g) \cong \mathrm{SU}(2)^g$ is an identification of topological spaces where $R(H^g)$ inherits a smooth structure from $\mathrm{SU}(2)^g \cong (S^3)^g$. The subspace $R(H^g) - S(H^g)$ is open, i.e. a smooth open manifold of dimension $3g$. As a consequence the representation space $\widehat{R}_2(H^g)$ is a smooth open manifold of dimension $3g-3$ (cf. [Sav99], Ch.14.3).

2.3.2 The representation space of $\pi_1(F^g)$

Since the fundamental groups $\pi_1(F^g)$ are trivial or abelian for $g=0$ or $g=1$ respectively, there are no irreducible representations. In all other cases we have the following statement for the irreducible part of the representation spaces.

Lemma 2.3.3 ([Igu50], [Sho36]). *Let F^g be the closed oriented surface of genus $g \geq 2$. Then $\widehat{R}(F^g)$ is a smooth manifold of dimension $6g-6$ (cf. [Sav99], Th.14.2 and Cor.14.3).*

It should be mentioned that the proof can be simplified by applying the Fox calculus (compare the proof of Lem.3.1 in [Heu98]).

2.3.3 The pillow case

Although the irreducible part of the representation space of the torus $F^1 = T^2$ is trivial, the abelian but non central part $R(T^2) - C(T^2) =: R^{nc}(T^2)$ forms a 2-dimensional manifold which becomes important in the sequel.

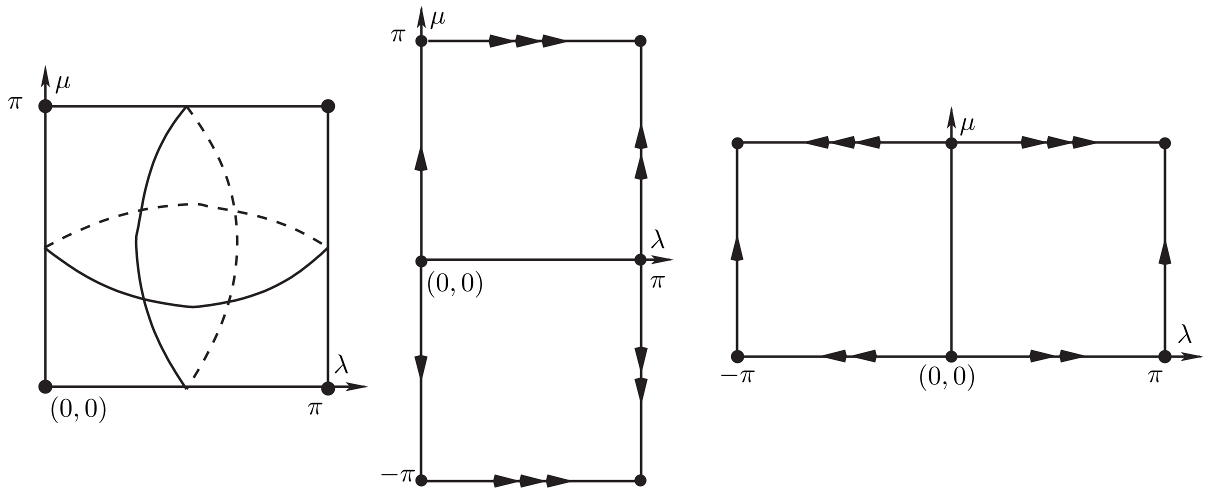


Figure 2.5: The pillow case and its two canonical parameterizations.

Lemma 2.3.4. *Let $F^1 = T^2$ be the 2-dimensional torus. Then $\widehat{R}(T^2) := R^{nc}(T^2)/\mathrm{SO}(3)$ is a pillow case ($\mathcal{PC}$), i.e. a 2-sphere with four cone points deleted.*

Proof. Let $\rho : \pi_1(T^2) \to \mathrm{SU}(2)$ be any non central representation. Because $\pi_1(T^2) = \mathbb{Z} \oplus \mathbb{Z}$ is an abelian group we have $\rho(l)\rho(m) = \rho(m)\rho(l)$ for matrices which represent the meridian m and the longitude l generating $\pi_1(T^2)$. Since the group U (1) $\subset$ SU(2) and its conjugates, $C \circ$ U (1), $C \in \mathrm{SU}(2)$ (for this notation see Ch.2.5), are the maximal commutative subgroups of SU(2) ([Sav99], Th.13.1) we obtain modulo SU(2)-conjugation (see Convention 2.5.1):

$$L = \rho(l) = \begin{pmatrix} e^{i\lambda} & 0 \\ 0 & e^{-i\lambda} \end{pmatrix} \quad \text{and} \quad M = \rho(m) = \begin{pmatrix} e^{i\mu} & 0 \\ 0 & e^{-i\mu} \end{pmatrix} .$$

Usage of the quaternion language (see Ch.2.5) tells us that it is possible to rotate $L = (\lambda, \boldsymbol{e}_x)$ within its conjugation class into $(\lambda, -\boldsymbol{e}_x) = (-\lambda, \boldsymbol{e}_x)$. (For example we can rotate by π with respect to axis $\boldsymbol{e}_y$ which corresponds to a conjugation with $C = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right) \in \mathrm{SU}(2)$.) As a consequence we can choose $\lambda \in [0, \pi]$. If $\lambda \in \{0, \pi\}$ then a conjugation of this kind is possible for $M = (\mu, \boldsymbol{e}_x)$ and we have $\mu \in [0, \pi]$ in these cases as well. This completes the proof. ■

Remark 2.3.5.

1. In general we identify a representation (L, M) of $\pi_1(T^2)$ with the corresponding pair of angles $(\lambda, \mu) \in ((0, \pi) \times [-\pi, \pi]) \cup (\{0\} \times (-\pi, \pi)) \cup (\{\pi\} \times (-\pi, \pi))$.
2. The (non-trivial) fundamental action on the universal covering of the pillow case (which is given by SU(2)-conjugation) takes a representation $(\lambda, \mu) \in [0, \pi] \times [-\pi, 0]$ into the representation we have for the other canonical parameterization of the pillow case: $(\lambda, \mu) \mapsto (-\lambda, -\mu) \in [-\pi, 0] \times [0, \pi]$. Normally we shall use the parameterization under 1.

2.3.4 Local properties of representation spaces

Since the space Hom $(G, \mathrm{SU}(2))$ is not a smooth manifold in general, we need the more general concept of the Zariski tangent space (cf. [Sha77], Ch.2). The Zariski tangent space equals the usual tangent space at an irreducible representation.

But the concept also works for a reducible point (see [Kla91], Ch.II). The idea of the Zariski tangent space refers to an observation of A. Weil situated in the context of the cohomology of groups (cf. [Wei64]). Within this context the system of equations defining the Zariski tangent vectors at $\rho \in$ Hom $(G, \mathrm{SU}(2))$ is same as the system of equations defining the space of 1-cocycles $Z^1_\rho(G, su(2))$ of G with coefficients in $su(2)$. Here the Lie algebra $su(2)$ is viewed as a G-module where the module structure is given by the composition $\tilde{\rho} : G \to$ Aut $(su(2))$, $\tilde{\rho}(g) = \rho(g)\circ$ (where $\circ$ is the conjugation action defined in Ch.2.5). Moreover, the 1-coboundaries $B^1_\rho(G, su(2))$ are induced by SU(2)-action and we obtain the Zariski tangent space of the SO(3)-equivalence classes at ρ as elements of the first cohomology group of G (with coefficients in the G-module $su(2)$, see [Sav99], Ch.15):

$$T_\rho \text{ Hom } (G, \mathrm{SU}(2))/\mathrm{SO}(3) = Z^1_\rho(G, su(2))/B^1_\rho(G, su(2)) = H^1_\rho(G, su(2)) \, .$$

Theorem 2.3.6 ([LM85], Ch.3.7). *Let $w = w(\mathbf{X})$, $\mathbf{X} \in \mathrm{SU}(2)^n$, be a word in* SU(2). *Then there is the following commutative diagram for the vectors of the Zariski tangent space:*

$$\begin{array}{ccc} T_{\mathbf{X}}\mathrm{SU}(2)^n & \xrightarrow{\cong} & su(2)^n \\ {\scriptstyle D_{\mathbf{X}}w}\downarrow & & \downarrow{\scriptstyle \mathrm{d}_{\mathbf{X}}w} \\ T_w\mathrm{SU}(2) & \xrightarrow{\cong} & su(2) \end{array} \tag{2.5}$$

The horizontal isomorphisms are given by right multiplication with $\mathbf{X^{-1}}$ and $w(\mathbf{X})^{-1}$ respectively; the derivation $\mathrm{d}_{\mathbf{X}}w = \sum_i \partial w/\partial X_{i|\mathbf{X}} \circ \mathrm{d}X_i$, $\mathrm{d}X_i := \mathrm{d}_{\mathbf{X}}X_i \in su(2)$, is provided by the Fox differential calculus (see [BZ85], Ch.9 B, for details).

Corollary 2.3.7. *Let $w = \prod_i X_i$, $X_i = 1 \in \mathrm{SU}(2)$, be a word in* SU(2). *Then $\mathrm{d}_{\mathbf{X}}w = \sum_i \mathrm{d}X_i$.*

Proof. Because $\partial w/\partial X_{i|\mathbf{X}}$ is central for all i we have $\partial w/\partial X_{i|\mathbf{X}} \circ \mathrm{d}X_i = \mathrm{d}X_i$ and the statement follows from diagram (2.5). ■

2.4 The Casson invariant (Sketch of definition)

In spring of 1985 Andrew Casson defined an integer invariant $\lambda(\Sigma)$ for homology 3-spheres Σ which initiated a remarkable progress in 3-dimensional topology. One important corollary states that the famous Rohlin invariant $\mu(\Sigma) \in \mathbb{Z}_2$ ([Roh52]) is not able to detect a counterexample to the Poincare conjecture. This is immediate because $\mu(\Sigma) \equiv \lambda(\Sigma)$ mod (2) holds and $\lambda(\Sigma)$ equals zero for a homotopy 3-sphere ([AM90], Introduction). Another no less important corollary deals with property P of a knot $k \subset S^3$. Here a knot is said to have property P if no non-trivial $\frac{1}{n}$-surgery along k yields a counterexample to the Poincare conjecture. The corollary states that if $\lambda(S^3 + \frac{1}{n+1}k) - \lambda(S^3 + \frac{1}{n}k) \neq 0$ then the knot k has property P. In the following we give a short survey of the definition of the Casson invariant. References are the books of Akbulut/McCarthy [AM90] and Saveliev [Sav99].

The definition of the Casson invariant consists of two parts. One deals with the computation of $\lambda(\Sigma)$, the other and more difficult part with the existence of the invariant.
The Casson invariant can be computed by a surgery formula which is based on a surgery description of Σ given by a framed link L in S^3 (see Th.2.1.3). It describes how $\lambda(\Sigma)$ changes if Σ is manipulated by an additional Dehn twist along one component k of L. Together with the starting value $\lambda(S^3) = 0$ this determines the invariant. Explicitly we obtain the formula:

$$\lambda'(k) := \lambda(\Sigma + \frac{1}{n+1}k) - \lambda(\Sigma + \frac{1}{n}k) = \frac{1}{2}\Delta''_{k\subset\Sigma}(1)\lambda'(k_{3_1}) = \pm\Delta''_{k\subset\Sigma}(1)\ , \qquad (2.6)$$

As the difference is independent of n it turns out that $\lambda'(k)$ is a knot invariant. Let $k_{3_1} \subset S^3$ denote the trefoil. Then an explicit calculation shows $\lambda'(k_{3_1} \subset S^3) = \pm 1$ where the sign depends on the orientations of representation spaces. The representation spaces come in via the equivalent definition of $\lambda(k)$ as an intersection number of oriented representation spaces. This interpretation provides the framework for the existence proof of $\lambda(k)$. Now the definition is done as follows. Let

$$\Sigma = H_1^g \cup H_2^g$$

be a Heegaard decomposition of the homology 3-sphere Σ where the handlebodies H_i^g intersect in the surface F^g of genus g (where we can assume $g \geq 2$). Then the inclusions $F^g \hookrightarrow H_i^g \hookrightarrow \Sigma$ induce the following commutative diagram of fundamental groups

$$\begin{array}{ccccc} & & \pi_1({H_1}^g) & & \\ & \overset{i_{1*}}{\nearrow} & & \overset{j_{1*}}{\searrow} & \\ \pi_1(F^g) & & & & \pi_1(\Sigma) \\ & \underset{i_{2*}}{\searrow} & & \underset{j_{2*}}{\nearrow} & \\ & & \pi_1({H_2}^g) & & \end{array}$$

By applying the R-functor we obtain a commutative diagram of representation spaces

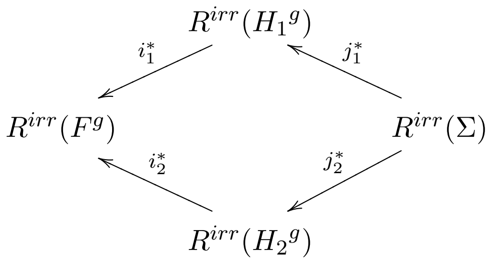
$$\begin{array}{ccccc} & & R({H_1}^g) & & \\ & \overset{i_1^*}{\swarrow} & & \overset{j_1^*}{\nwarrow} & \\ R(F^g) & & & & R(\Sigma) \\ & \underset{i_2^*}{\nwarrow} & & \underset{j_2^*}{\swarrow} & \\ & & R({H_2}^g) & & \end{array}$$

where all maps are injective. The following two statements, both of which are based on the fact that Σ is a homology sphere, are essential for the definition of $\lambda(\Sigma)$ as an intersection number.

Lemma 2.4.1. *Let Σ be a homology 3-sphere. Then any abelian representation $\rho : \pi_1(\Sigma) \to \mathrm{SU}(2)$ is trivial ([Sav99], L.14.1).*

Lemma 2.4.2. *The intersection $R(H_1^g) \cap R(H_2^g) \subset R(F^g)$ is transversal at the trivial representation in $R(F^g)$ ([Sav99], L.16.1).*

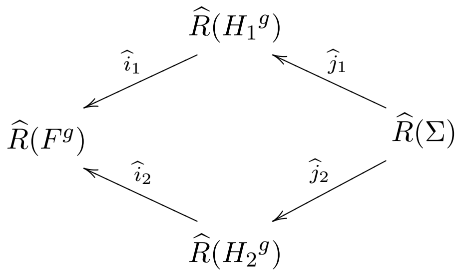
The restriction to irreducible representations yields the following commutative diagram of inclusions



An immediate consequence of the lemmata 2.4.1 and 2.4.2 is

Theorem 2.4.3. *a) The trivial representation of* $R(\Sigma)$ *is isolated.*
b) The intersection of the smooth open manifolds $R^{irr}(H_i^g)$, $i = 1, 2$, *in* $R^{irr}(F^g)$ *is compact.*

Taking the SO(3)-quotient of the representation spaces in the last diagram we obtain the commutative diagram of embeddings



Remark 2.4.4. Using the Heegaard diagram's point of view to describe the splitting of Σ we regard $\widehat{R}(H_1^g) =: \widehat{R}_1 \subset \widehat{R}(F^g)$ as trivially embedded. The embedding $\widehat{h}(\widehat{R}(H_2^g)) \subset \widehat{R}(F^g)$ is still denoted by $\widehat{R}(H_2^g) =: \widehat{R}_2$.

As theorem 2.4.3 holds for the quotient spaces as well ([Sav99], Cor.16.3) we can isotope $\widehat{R}_1$ into $\widetilde{R}_1$ with compact support where $\widetilde{R}_1$ is transversal to $\widehat{R}_1$. Moreover, the manifolds have dimensions

$$\dim \widehat{R}_i = 3g - 3 = \frac{1}{2} \dim \widehat{R}(F^g)$$

so that the intersection $\widetilde{R}_1 \pitchfork \widehat{R}_2 \subset \widehat{R}(F^g)$ is a finite number of points. By an orientation of $\widehat{R}_i$, $i = 1, 2$, the algebraic intersection number

$$\#\widehat{R}_1 \cap \widehat{R}_2 := \sum_{p \in \widetilde{R}_1 \pitchfork \widehat{R}_2} \varepsilon_p \ , \ \varepsilon_p \in \pm 1$$

is defined and independent of the chosen isotopy due to theorem 2.4.3.

Definition 2.4.5. *Given a genus g Heegaard splitting* $\Sigma = H_1^g \cup H_2^g$ *of a homology-3-sphere, its Casson invariant is*

$$\lambda(\Sigma) = \frac{(-1)^g}{2} \#\widehat{R}_1 \cap \widehat{R}_2$$

It turns out that the intersection number is a well defined invariant. The factor $(-1)^g$ is necessary to ensure the independence of the chosen Heegaard splitting (cf. [Sav99], Ch.16.3). That $\lambda(\Sigma)$ is always an integer is not immediately clear from the definition. But it is known that the intersection number defined above turns out to be even. The latter is shown within the existence proof of $\lambda(\Sigma)$ and in the end is a consequence of the 2-fold covering $SU(2) \to SO(3)$ (compare Sec. 4.3 and [AM90], Cor.4.5).

2.5 Quaternions

Sometimes it is more convenient to work with quaternions instead of SU(2)-matrices. Therefore we identify SU(2) with the unit quaternions $\mathbb{H}_1 \subset \mathbb{H}$ where the isomorphism is given by

$$\begin{pmatrix} a & b \\ -\bar{b} & \bar{a} \end{pmatrix} \mapsto a + \mathbf{j}b \, .$$

The Lie algebra of $\mathbb{H}_1$ is the set $su(2) \cong \mathbb{R}^3$ of pure quaternions and $\mathbb{H}_1$ acts via conjugation $\circ$ on $su(2)$, i.e. $Q \circ A := QAQ^{-1} \in su(2)$ for $Q \in \mathbb{H}_1$ and $A \in su(2)$. The set of pure unit quaternions $su(2) \cap \mathbb{H}_1$, which is homeomorphic to the 2-sphere S^2, will be identified with S^2. More generally, we consider the argument function $\arg : \mathrm{SU}(2) \twoheadrightarrow [0, \pi]$ given by $\arg(Q) = \arccos(\mathrm{tr}\,(Q)/2)$. For $\alpha \in (0, \pi)$ we have: $S^2_\alpha := \arg^{-1}(\alpha)$ is a 2-sphere and $S^2 = S^2_{\pi/2}$.

For each quaternion $Q \in \mathbb{H}_1$ there is an angle $\alpha \in [0, \pi]$ and a vector $\boldsymbol{q} \in S^2$ such that $Q = \cos\alpha + \sin\alpha\boldsymbol{q}$ (where the coefficients of $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$ in S^2 are given by the coefficients of $\mathbf{i}$, $\mathbf{j}$, $\mathbf{k}$ resp.). The pair $(\alpha, \boldsymbol{q})$ is unique if and only if $Q \neq \pm 1$. We will write $(\alpha, \boldsymbol{q})$ shorthand for $\cos\alpha + \sin\alpha\boldsymbol{q}$ and call α the *angle of the quaternion* Q and $\boldsymbol{q}$ its *spatial part*.
Given two unit quaternions $P, Q \in \mathbb{H}_1$ there is a formula for the product

$$PQ = (\alpha, \boldsymbol{p})(\beta, \boldsymbol{q}) = \begin{pmatrix} \cos\alpha\cos\beta - \sin\alpha\sin\beta\,\boldsymbol{p}\cdot\boldsymbol{q} \\ \cos\alpha\sin\beta\boldsymbol{q} + \cos\beta\sin\alpha\boldsymbol{p} + \sin\alpha\sin\beta\boldsymbol{p}\times\boldsymbol{q} \end{pmatrix} \tag{2.7}$$

where $\boldsymbol{p}\cdot\boldsymbol{q}$ denotes the dot and $\boldsymbol{p}\times\boldsymbol{q}$ the vector product of $\boldsymbol{p}$ and $\boldsymbol{q}$ in $\mathbb{R}^3$.

Convention 2.5.1. 1. *Let ρ denote a* SU(2)*-representation of a finitely generated group G. Then the capitals indicate the equally named elements of the represented group, i.e. $X = \rho(x)$ for $x \in G$ and $\rho \in R(G)$.*

2. *Let $\prod_{i=1}^n X_i$ be a product of* SU(2)*-matrices. Then the spatial part of the product is denoted by $\boldsymbol{x}_1\boldsymbol{x}_2\ldots\boldsymbol{x}_n$.*

From the commutativity of matrices under the trace function we see from equation (2.7) that the dot product of the spatial parts of two given quaternions is invariant under conjugation. This can also be deduced from the following important geometrical interpretation of conjugation.

Lemma 2.5.2. *Let $P = (\beta, \boldsymbol{p})$ and $Q = (\alpha, \boldsymbol{q}) \in \mathbb{H}_1$ be unit quaternions. Then $Q \circ P$ rotates the spatial part $\boldsymbol{p}$ of P by angle 2α with respect to axis $\boldsymbol{q} \in S^2$ ([Sav99], Th.13.4).*

Corollary 2.5.3. *The homomorphism $\circ : \mathrm{SU}(2) \to \mathrm{SO}(3)$, $Q \mapsto Q\circ \in \mathrm{SO}(3) \subset \mathrm{Aut}\,(su(2)) = \mathrm{Aut}\,(\mathbb{R}^3)$ is a well defined Lie group homomorphism. It is the universal 2-fold cover of* SO(3).

For $\mathbf{e}^{i\alpha} := \begin{pmatrix} e^{i\alpha} & 0 \\ 0 & e^{-i\alpha} \end{pmatrix} = (\alpha, \boldsymbol{e}_x) \in \mathrm{SU}(2)$ we identify the tangent space $T_{\mathbf{e}^{i\alpha}}(S^2_\alpha) = \mathrm{span}\,(\mathbf{j}, \mathbf{k})$ with the complex plane $\mathbb{C}$ via the multiplication by $-\mathbf{j} \in \mathbb{H}_1$. From lemma 2.5.2 follows

Remark 2.5.4. Under the identification $T_{e^{i\alpha}}(S^2_\alpha) = \mathrm{span}\,(\mathbf{j}, \mathbf{k}) = \mathbb{C}$ the action of $\mathbf{e}^{i\alpha}\circ$ transforms into a rotation by angle 2α in $\mathbb{C}$, i.e. a multiplication by $e^{2i\alpha}$.
This is a very useful fact because together with equation (2.5) it follows that we are able to calculate the Zariski tangent space at $\rho_\alpha \in S^\alpha(S^3 - k)$ from an Alexander matrix $A_k(t)$ of k by setting $t = e^{2i\alpha}$ (where α denotes that the meridians of k are represented by SU(2)-matrices with trace $2\cos\alpha$, cf. [Kla91], Ch.2). Moreover, we will use this correlation in a similar way to obtain information about the reducibles of $R(\Sigma - k)$.

In the sequel we will make frequent use of the following obvious identities ($\cos\alpha =: c_\alpha$, $\sin\alpha =: s_\alpha$).

- For the inverse of $q = (\alpha, \boldsymbol{q}) \in \mathbb{H}_1$ holds

$$q^{-1} = \bar{q} = c_\alpha - s_\alpha \boldsymbol{q} = (-\alpha, \boldsymbol{q}) \ . \tag{2.8}$$

- Let $q = (\alpha, \boldsymbol{q})$ and $\tilde{q} = (-\alpha, -\boldsymbol{q})$ in $\mathbb{H}_1$ be given. Then

$$q = c_\alpha + s_\alpha \boldsymbol{q} = c_{-\alpha} - s_{-\alpha} \boldsymbol{q} = \tilde{q} \ . \tag{2.9}$$

- For the negative of $q = (\alpha, \boldsymbol{q}) \in \mathbb{H}_1$ we have

$$-q = -(\alpha, \boldsymbol{q}) = -c_\alpha - s_\alpha \boldsymbol{q} = c_{\pi+\alpha} + s_{\pi+\alpha} \boldsymbol{q} = (\pi + \alpha, \boldsymbol{q}) \ . \tag{2.10}$$

Chapter 3

The definition of $s^\alpha(k \subset \Sigma)$

3.1 The definition of the standard position for $k' \subset H_1^g$

Let k be a knot in a homology sphere Σ and (H_1^g, h) a Heegaard diagram associated with a Heegaard splitting of Σ. In order to define $s^\alpha(k \subset \Sigma)$ as an intersection number of representation spaces we have to determine the representation space $\widehat{R}(H_2^g) =: \widehat{R}_2$ and the representation space of the complement of k' in H_1^g, i.e. $\widehat{R}(H_1^g - k')$. (As in the case of the Casson invariant we assume $g \geq 2$. The only non-trivial case for a Heegaard splitting of genus 1, the $(2,n)$-torus knots in S^3, is discussed in section 4.2.) The intersection occurs in the representation space $\widehat{R}(F^g)$ of the common boundary F^g of the handlebodies. In order to define a Casson-Lin invariant for the knot complement as an intersection number of representation spaces one has to fix the holonomy around k (which means to represent all knot meridians by SU(2)-matrices with a fixed trace $2\cos\alpha$).

Definition 3.1.1. *Let M be a closed 3-manifold and k a knot in M and let $R(M-k)$ denote the representation space of the fundamental group of $M-k$. Then $R^\alpha(M-k)$ denotes the subspace of all representations which assign* SU(2)*-matrices with a fixed trace $2\cos\alpha$ to all meridians of k.*

The definition of the intersection number $s^\alpha(k \subset \Sigma)$ uses a refined Heegaard diagram $(H_1^{g''}, h'')$ which is obtained from the given decomposition by stabilizations of the handlebodies together with isotopies of the knot. The resulting *standard position* for k' is a plat presentation of k' which respects the first chosen splitting. To work with a plat presentation seems to be natural since the map $\pi_1(F^{g''}) \to \pi_1(H_1^{g''})$ induced by inclusion is surjective in this case.

Let F be a Seifert surface of k and establish $F \subset H_1^g$ by an isotopy (see Th.2.2.8). Consider F as a disc with (thin) bands attached and let m, $0 \leq m \leq g$, be the number of handles of H_1^g without bands inside. Furthermore denote a regular projection for $k' \subset H_1^g \subset S^3$ by k'_p. Then perform the following manipulations:

1. "Comb" the $g-m$ handles having bands of F inside, so that all the crossings of k'_p lie entirely inside the projection disc D^2 of $B^3 \subset H_1^g$ (see Fig. 3.1).

2. If there are t bands passing through a handle, drill $t-1$ holes to get t handles with exactly one band inside. Slide the new handles along the boundary of B^3 such that both ends lie side by side on ∂B^3. Regarding the 2 strands as 1 band and $t-1$ bands respectively, figure 3.1 shows one step of the inductive process. Then apply the procedure described in the proof of theorem 2.2.7 to obtain a homologically flat Heegaard diagram $(H_1^{g'}, h')$ with $H_1^{g'}$ of genus $g' := s+m$. This presentation is called a *single band presentation* for k'.

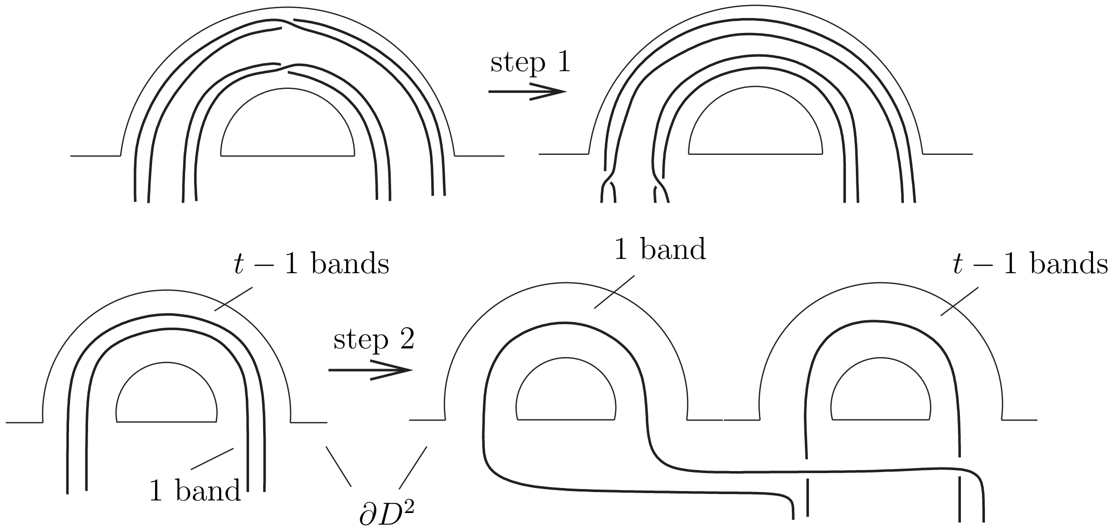


Figure 3.1: Establishing the standard position: Step 1 and 2.

3. Drill each of the s handles with bands inside one time to obtain the *single strand presentation* for k' (regarding the bands as strands of k' this corresponds to step 2 in figure 3.1).

4. Isotope the knot to obtain a $2n$-plat presentation $\widehat{\beta}$ of k', $n := 2s + r$. This presentation follows from a $2n$-braid $\beta \in \mathcal{B}_{2n}$ by closing it with $2n$ simple arcs. Here $\mathcal{B}_{2n}$ denotes the braid group with $2n - 1$ generators (see Fig.3.2). Then stabilize the Heegaard diagram to obtain a single strand presentation of the knot where all upper closing arcs pass through exactly one handle. The Heegaard splitting of this *standard position* has genus $g'' := n + m$.

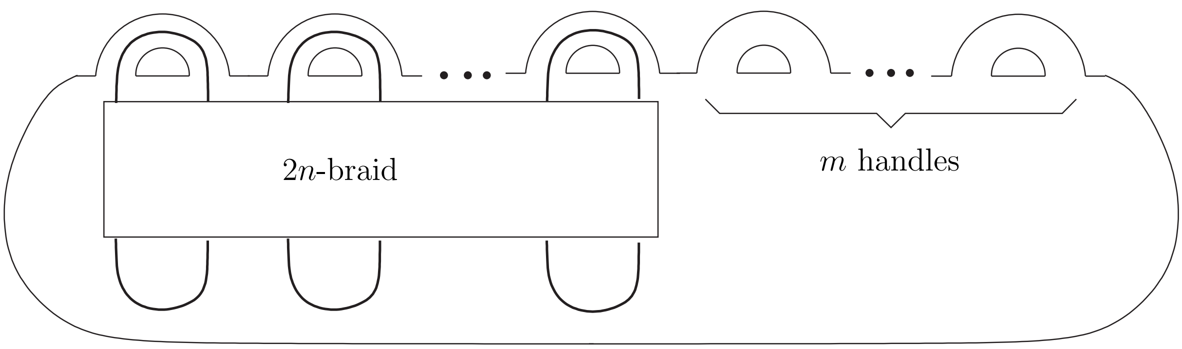


Figure 3.2: The standard position of $k' \subset H_1^{g''}$.

Definition 3.1.2. *Let k be a knot in a homology sphere Σ. The sequence of stabilizations and isotopies described above leads to a new Heegaard splitting $\Sigma = H_1^{g''} \cup H_2^{g''}$ with $k \subset H_1^{g''}$. Then we call a Heegaard diagram $(H_1^{g''}, h'')$ associated with this splitting a* standard diagram *and $k' \subset H_1^{g''} \subset S^3$ being in* standard position. *Further we denote the $2n$-braid associated with the plat presentation of k' by β.*

Remark 3.1.3. 1. If not stated otherwise, let a Heegaard diagram (H_1^g, h) of Σ with $k' \subset H_1^g$ and $g \geq n$ always be standard.

2. The proof of theorem 3.2.5 shows that a standard diagram is actually homologically flat.

3.2 The representation space of $\pi_1(H_1^g - k')$

Let $k' \subset H_1^g \subset S^3$ in standard position be given by the plat presentation $\widehat{\beta}$. Then $\pi_1(H_1^g - k')$ is generated by the g longitudes l_i of H_1^g and the n knot meridians x_{2i}, $1 \leq i \leq n$, corresponding to the upper closing arcs of $\widehat{\beta}$.

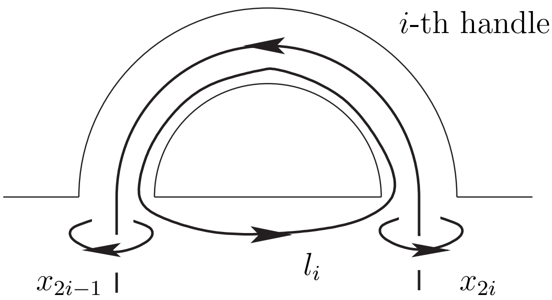


Figure 3.3: The orientations of the longitudes and knot meridians.

To figure out the relations for the generators we have to orient them. Hence we fix an orientation of k' and orient the meridians of k' by the "right hand rule" (where the thumb of the right hand points in the direction of k and the fingers give the orientation of the meridians). For the longitudes we choose the standard orientation (see Fig.3.3). Let x_{2i} denote the knot meridian at the attaching disc, where the knot runs into the i-th handle (with respect to the orientation of l_i). Then the meridian at the disc where the knot leaves this handle is given by conjugation with l_i, i.e. $x_{2i-1} = l_i \circ x_{2i}$, $1 \leq i \leq n$. Since the path enclosing the $2n$-plat $\widehat{\beta}$ is contractible in $H_1^g - k'$ we obtain the further relation $\prod_{i=1}^{n} x_{2i-1}^{-\varepsilon_i} x_{2i}^{\varepsilon_i} = 1$ for the knot meridians. Here $\varepsilon_i = 1$ ($\varepsilon_i = -1$) holds if k' and l_i are parallel (anti-parallel).

Let $\beta(x_i)$, $1 \leq i \leq 2n$, denote the images of the upper knot meridians under the braid automorphism induced by β. Then the lower closing arcs contribute the relations $\beta(x_{2i-1}) = \beta(x_{2i})$, $1 \leq i \leq n$.

Choosing the longitudes l_i and the corresponding handle meridians m_i of H_1^g as generators of $F^g = \partial H_1^g$, the relation $\prod_{i=1}^{g}[l_i, m_i] = 1$ is satisfied in $\pi_1(F^g)$. Because $m_i = 1$, $n+1 \leq i \leq g$, holds in $\pi_1(H_1^g - k')$ we obtain the additional relation $\prod_{i=1}^{n}[l_i, x_{2i}^{\varepsilon_i}] = 1$. Note that the exponents are compatible with the orientation conventions in chapter 3.3 provided that we chose a suitable orientation of F^g.
We summarize the result in the next lemma.

Lemma 3.2.1. *Let $k' \subset H_1^g \subset S^3$ be a knot in standard position be given by the plat presentation $\widehat{\beta}$. Then we have the following presentation of $\pi_1(H_1^g - k')$ in terms of the longitudes l_i, $1 \leq i \leq g$, of H_1^g and the knot meridians x_{2i}, $1 \leq i \leq n$ of the handles passed through by k':*

$$\pi_1(H_1^g - k') = \langle l_1, \ldots, l_g, x_2, \ldots, x_{2n} | \\ x_{2i-1} = l_i \circ x_{2i} \, , \; \beta(x_{2i-1}) = \beta(x_{2i}), 1 \leq i \leq n \, , \; \prod_{i=1}^{n}[l_i, x_{2i}^{\varepsilon_i}] = 1 \, , \; \prod_{i=1}^{n} x_{2i-1}^{-\varepsilon_i} x_{2i}^{\varepsilon_i} = 1 \, , \; \varepsilon_i = \pm 1 \rangle$$

According to Lin (cf. [Lin92], p.343) we want to recover the part of the representation space $\widehat{R}(H_1^g - k')$ which is relevant to the definition of $s^\alpha(k \subset \Sigma)$ from an intersection of two manifolds, the *diagonal* and the *graph*. The graph is determined by the diffeomorphism induced on the representation spaces by the braid automorphism. The definition of the manifolds is based on the following geometrical observation which is in connection with the plat presentation of $k' \subset H_1^g$. A $2n$-plat presentation $\widehat{\beta}$ of $k \subset S^3$ gives rise to a splitting

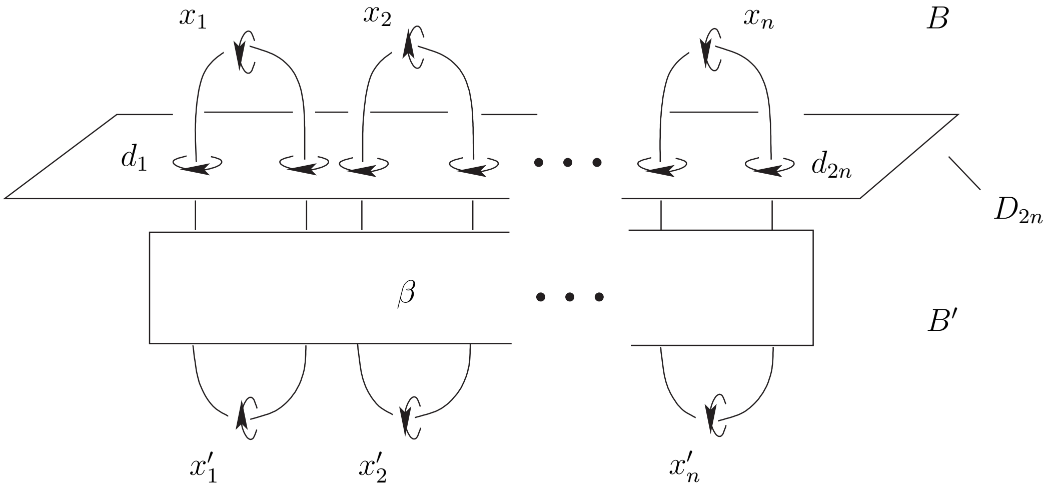


Figure 3.4: The decomposition $B \cup_{D_{2n}} B'$.

$$S^3 = B \cup_{D_{2n}} B'$$

where B and B' are handlebodies of genus n and $D_{2n} = B \cap B'$ is a 2-sphere with $2n$ holes (see Fig.3.4). Then the representation space of $\pi_1(S^3 - k)$ is given by the intersection of the embeddings of $\widehat{R}(B)$ and $\widehat{R}(B')$ in $\widehat{R}(D_{2n})$ (cf. [Heu98], Ch.3):

$$\widehat{R}(S^3 - k) = \widehat{R}(B) \cap \widehat{R}(B') \subset \widehat{R}(D_{2n}) \ .$$

Slight generalizations due to the additional relations in $\pi_1(H_1^g - k')$ lead to the desired manifolds.

Let $k' \subset H_1^g$ be a knot in standard position given by the $2n$-plat $\widehat{\beta}$. Then the counterpart of $\widehat{R}(D_{2n})$ is defined by

$$H^\alpha_{n,g} = \{(X_1, X_2, \ldots, X_{2n}, L_1, \ldots, L_g) \in (S^2_\alpha)^{2n} \times \mathrm{SU}(2)^g | \prod_{i=1}^{2n} X_i = 1\} \ .$$

It can be shown by the Fox calculus that the map

$$\begin{array}{rccc} f & : & (S^2_\alpha)^{2n} & \to & \mathrm{SU}(2) \\ & & (X_1, \ldots, X_{2n}) & \mapsto & \prod_{i=1}^{2n} X_i \end{array}$$

is regular at $1 \in \mathrm{SU}(2)$ for an irreducible set of matrices X_i (see [Heu98], L.3.1). Therefore $\widehat{H}^\alpha_{n,g}$ is a manifold with

$$\dim \widehat{H}^\alpha_{n,g} = 4n + 3g - 6$$

Since the intersection number $s^\alpha(k \subset \Sigma)$ is supposed to count *non-abelian* representations of the knot complement we may restrict our interests to the set

$$I^\alpha := \{(X_1, \ldots, X_n, L_1, \ldots, L_n) \in ((S^2_\alpha)^n \times \mathrm{SU}(2)^n)^{irr}\} .$$

Then the *diagonal*, being the counterpart of $\widehat{R}(B)$, is defined by

$$H^\alpha_{n,g} \supset \Lambda^\alpha_{n,g} = \{(L_1 \circ X_1^{-\varepsilon_1}, X_1^{\varepsilon_1}, \ldots, L_n \circ X_n^{-\varepsilon_n}, X_n^{\varepsilon_n}, L_1, \ldots, L_g)| \\ (X_1, \ldots, X_n, L_1, \ldots, L_n) \in I^\alpha \ , \ \prod_{i=1}^n [L_i, X_{2i}^{\varepsilon_i}] = 1 \ , \ L_j \in \mathrm{SU}(2) \ , \ n+1 \le j \le g \ , \ \varepsilon_i = \pm 1\} .$$

Lemma 3.2.2. *Let the map g be defined by*

$$\begin{array}{cccc} g & : & (S^2_\alpha)^n \times \mathrm{SU}(2)^n & \to & \mathrm{SU}(2) \\ & & (X_1, \ldots, X_n, L_1 \ldots, L_n) & \mapsto & \prod_{i=1}^n [L_i, X_{2i}^{\varepsilon_i}] \end{array} .$$

Then the irreducible part of $g^{-1}(1)$ is a manifold of dimension $5n-3$.

Proof. Because the arguments used in the proof of theorem 2.3.3 (cf. [Sav99], Th.14.2) involve only the spatial parts of the quaternions, they also apply for the case considered. Since the trace of the matrices X_i is fixed their dimensional contribution is only 2. This proves the statement. ∎

It follows from lemma 3.2.2 immediately that the diagonal is a manifold with

$$\dim \widehat{\Lambda}^\alpha_{n,g} = 5n - 3 + 3(g-n) - 3 = 2n + 3g - 6 .$$

Still denoting the diffeomorphism induced on the representation spaces by β, we define the *graph*

$$\begin{aligned} H^\alpha_{n,g} \supset \Gamma^\alpha_\beta \quad = \quad & \{(\beta^{-1}(X_1')^{\varepsilon_1'}, \beta^{-1}(X_1')^{-\varepsilon_1'}, \ldots, \beta^{-1}(X_n')^{\varepsilon_n'}, \beta^{-1}(X_n')^{-\varepsilon_n'}, \\ & L_1, \ldots, L_g) \in (S^2_\alpha)^n \times \mathrm{SU}(2)^g \ , \ \varepsilon_i' = \pm 1\} . \end{aligned}$$

The ε_i' depend on the permutation induced by the braid automorphism β. Thus they may differ from the ε_i in the relations of $\pi_1(H^g_1 - k')$. For the dimension of the graph holds

$$\dim \widehat{\Gamma}^\alpha_\beta = 2n + 3g - 3 .$$

The space of equivalence classes of representations of $\pi_1(H^g_1 - k')$ mapping the knot meridians to non-abelian matrices can be identified with the intersection of diagonal and graph:

$$\widehat{R}^\alpha_1(H^g_1 - k') \supset \widehat{\Lambda}^\alpha_{n,g} \cap \widehat{\Gamma}^\alpha_\beta \subset \widehat{H}^\alpha_{n,g} .$$

This leads to the definition of the manifold $\widehat{R}'^\alpha_1(\beta)$ in the following lemma.

Lemma 3.2.3. *Let the manifolds $\widehat{\Lambda}^\alpha_{n,g}$, $\widehat{\Gamma}^\alpha_\beta$ and $\widehat{H}^\alpha_{n,g}$ be given as before. Choose an isotopy $\widehat{\Gamma}^\alpha_\beta \rightsquigarrow \widetilde{\Gamma}^\alpha_\beta$ such that $\widehat{\Lambda}^\alpha_{n,g} \pitchfork \widetilde{\Gamma}^\alpha_\beta \subset \widehat{H}^\alpha_{n,g}$. Then*

$$\widehat{R}'^\alpha_1(\beta) := \widehat{\Lambda}^\alpha_{n,g} \pitchfork \widetilde{\Gamma}^\alpha_\beta \subset \widehat{H}^\alpha_{n,g}$$

is a manifold with

$$\dim \widehat{R}'^\alpha_1(\beta) = \dim \widehat{\Lambda}^\alpha_{n,g} + \dim \widehat{\Gamma}^\alpha_\beta - \dim \widehat{H}^\alpha_{n,g} = 3g - 3 .$$

Remark 3.2.4. Note that the manifold $\widehat{R}_1'^{\alpha}(\beta)$ apart from plat presentation $\widehat{\beta}$ also depends on the Heegaard splitting of Σ.

By fixing $\mathbf{L} = \mathbf{L}_0 \in \mathrm{SU}(2)^g$ at a point $p \in \widehat{\Lambda}^{\alpha}_{n,g} \pitchfork \widetilde{\Gamma}^{\alpha}_{\beta}$ we conclude that $\widehat{\Lambda}^{\alpha}_{n,g|\mathbf{L}_0} \pitchfork \widetilde{\Gamma}^{\alpha}_{\beta|\mathbf{L}_0}$ is 0-dimensional. Furthermore, since the map $i_{1*} : \pi_1(F^g) \to \pi_1(H_1^g - k')$ induced by inclusion is surjective, we obtain a proper map of representation spaces

$$\widehat{i}_1 : \widehat{\Lambda}^{\alpha}_{n,g} \to \widehat{R}(F^g) \ . \tag{3.1}$$

and therefore a proper embedding of manifolds (cf. [AM90], Cor.1.2 (c))

$$\widehat{i}_1 : \widehat{R}_1'^{\alpha}(\beta) \hookrightarrow \widehat{R}(F^g) \ , \tag{3.2}$$

where we identify $\widehat{i}_1(\widehat{R}_1'^{\alpha}(\beta))$ with $\widehat{R}_1'^{\alpha}(\beta)$ to simplify the notation.

Since we want to compute $s^{\alpha}(k \subset \Sigma)$ by observing its behavior under crossing changes we need a more general version of theorem 2.2.8.

Theorem 3.2.5. *Let (H_1^g, h) be a standard Heegaard diagram for $k \subset \Sigma$ and let $k' \subset S^3$ be the corresponding knot in S^3. Then*

$$\Delta_{k\subset\Sigma}(t) = \Delta_{k'\subset S^3}(t)$$

holds for all knots derived from k' by crossing changes.

Proof. The standard position for k' is obtained from the single band presentation (step 2) by stabilization of all s handles, for which the bands pass through. From this, the standard presentation of k' follows by further (trivial) stabilizations. Let l_i, $1 \le i \le g$, denote the standard longitudes generating $\pi_1(H_1^g)$ where the l_i, $1 \le i \le n$, denote the handles, for which the knot in standard position passes through. The gluing curves corresponding to the holes we drill to establish the single strand presentation (step 3) contribute the following relations to the Heegaard diagram (see Fig.3.5):

$$l_{2i}^{-1} m_{2i-1}^{-1} l_{2i-1}^{-1} m_{2i-1} = 1 \ \ , \ \ 1 \le i \le s \ .$$

Further we have to substitute each meridian $m_i = x_{2i}^{\pm 1} x_{2i-1}^{\mp 1}$ appearing in the relations of the single band Heegaard diagram by $m_{2i-1} m_{2i}$ where $m_{2i-1} = x_{2i-1}^{-\varepsilon_i}$ and $m_{2i} = x_{2i}^{\varepsilon_i}$, $1 \le i \le s$. The remaining r relations contributed by step 4 are trivial, i.e. $l_i = 1$, $2s+1 \le i \le n$.

The Alexander polynomial of $k \subset \Sigma$ can be obtained as the first principal minor of the matrix of the abelianized Fox derivatives of the relations determining the knot group $\pi_1(\Sigma - k)$ (cf. the construction in [HPS01], Sec.2.1). In our case we use the relations of $\pi_1(H_1^g - k')$ and the additional relations associated with the g gluing curves $h(\partial D_i)$ (which implies the relation $\prod_{i=1}^n [l_i, x_{2i}^{\varepsilon_i}] = 1$ in $\pi_1(H_1^g - k')$).

Let γ_i, $1 \le i \le n+g$, denote the generators of $\pi_1(H_1^g - k')$ corresponding to the $2n$-plat presentation $\widehat{\beta}$ of k' with $\gamma_i = l_i$, $1 \le i \le g$, and $\gamma_i = x_{2i}$, $g+1 \le i \le n+g$. Further let R_i, $1 \le i \le g$, be the relations associated with the g gluing curves and R_i, $g+1 \le i \le n+g$, the n relations associated with the braid automorphism. Then abelianizing the Fox derivations $\partial R_i / \partial \gamma_j$ by $\mathrm{ab}(l_i) = 1$ and $\mathrm{ab}(x_i) = t$ yields

$$\left(\frac{\partial R_i}{\partial \gamma_j}\right)_{\mathrm{ab}} = \left(\begin{array}{ccc|c} E_s & 0 & 0 & 0 \\ E_s & E_s & 0 & 0 \\ 0 & 0 & E_{g-2s} & 0 \\ \hline & * & & \left(\frac{\partial R_{i \ge g+1}}{\partial x_j}\right)_{\mathrm{ab}} \end{array}\right) \tag{3.3}$$

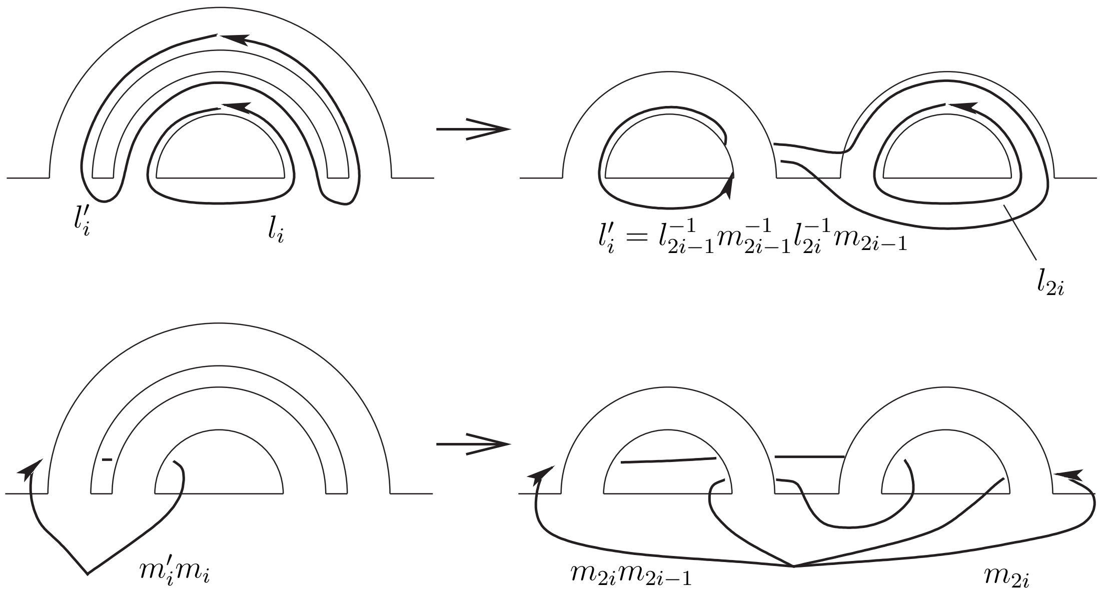


Figure 3.5: The gluing curves after drilling and moving a handle (step 3).

where E_k denotes an $k \times k$-identity matrix.
Since every appearance of a band meridian in a relation contributes $\mathrm{ab}(m_{2k}m_{2k-1}) = \mathrm{ab}(x_{2k}^{\pm 1} x_{2k-1}^{\mp 1}) = 1$ the results for the derivatives with respect to the longitudes follow immediately from corollary 2.3.7. The equalities $\mathrm{ab}(\partial R_i / \partial \gamma_j) = 0$, $1 \leq i \leq g$, $g+1 \leq j \leq n+g$, hold because we use a flat Heegaard diagram. The latter implies that the derivatives with respect to the knot meridians x_{2k-1} and x_{2k} respectively cancel out. Because $\left(\frac{\partial R_{i \geq g+1}}{\partial x_j}\right)_{\mathrm{ab}}$ is an Alexander matrix of $k' \subset S^3$ the statement follows. ∎

Theorem 3.2.5 has some immediate corollaries:

Corollary 3.2.6. *Let $k \subset \Sigma$ be a knot in a homology 3-sphere. Then there exists a knot $k_0 \subset \Sigma$ with trivial Alexander polynomial $\Delta_{k_0 \subset \Sigma}(t) = 1$ (or trivial signature $\sigma_{k_0 \subset \Sigma}(\omega) = 1$) in the same homotopy class as k.*

Proof. With $k' \subset H_1^g \subset S^3$ from theorem 2.2.8 we obtain $\Delta_{k \subset \Sigma}(t) = \Delta_{k' \subset S^3}(t)$ (or $\sigma_{k \subset \Sigma}(\omega) = \sigma_{k' \subset S^3}(\omega)$ resp.). Let a regular projection and k' as a map $k' : [0,1] \to S^3$, $r \mapsto k'(r)$, with $k'(0) = k'(1)$ be given. Then we are able to change the crossings of k' such that the image $k'(r)$ is, with respect to the coordinate perpendicular to the projecting plane, a monotone curve for $r \in (0, 1-\varepsilon)$, $\varepsilon > 0$. Joining the points $k'(1-\varepsilon)$ and $k'(0)$ by a straight line yields the trivial knot $k_0' \subset S^3$. From theorem 3.2.5 follows that identical changes of crossings for $k \subset H_1^g \subset \Sigma$ lead to the aspired $k_0 \subset \Sigma$. ∎

Remark 3.2.7. Corollary 3.2.6 implies that in each homotopy class of a knot $k \subset \Sigma$ all possible Alexander polynomials are realized.

Corollary 3.2.8. *Let $k_0 \subset \Sigma$ be a knot in a homology sphere with trivial Alexander polynomial. Then all possible Alexander polynomials $\Delta(t)$ are realized by k_0 with one crossing changed.*

Proof. The band construction used by Kondo to establish the statement for knots in S^3 (see [Kon78]) can also be applied for the $k' \subset H_1^g \subset S^3$ of theorem 2.2.8. ■

Remark 3.2.9. 1. To simplify the notation we do not indicate the homology 3-sphere Σ if any confusion can be ruled out.

2. A comparison of the proofs given in [AR99] or [Les98], Pr.4.6, and the proof of corollary 3.2.6 shows the advantages of using a Heegaard splitting of Σ as a starting point.

3.3 Orientations of the manifolds

Let $\Sigma = H_1^g \cup H_2^g$ be a Heegaard splitting of genus g. The inclusions $i_k : F^g \to H_k^g$, $k = 1, 2$, induce homomorphisms $i_k^\# : H^1(H_k^g, \mathbb{R}) \to H^1(F^g, \mathbb{R})$. The Mayer-Vietoris exact sequence

$$\cdots \to H^1(\Sigma, \mathbb{R}) \to H^1(H_1^g, \mathbb{R}) \oplus H^1(H_2^g, \mathbb{R}) \overset{i_1^\# \oplus i_2^\#}{\to} H^1(F^g, \mathbb{R}) \to H^2(\Sigma, \mathbb{R}) \to \cdots$$

with $H^1(\Sigma, \mathbb{R}) = H^2(\Sigma, \mathbb{R}) = 0$ implies $i_1^\# H^1(H_1^g, \mathbb{R}) \oplus i_2^\# H^1(H_2^g, \mathbb{R}) = H^1(F^g, \mathbb{R}) \cong \mathbb{R}^{2g}$. Then orienting $H^1(F^g, \mathbb{R})$ by a symplectic basis (which is canonical, provided an orientation of F^g is given, cf. [Sav99], Lem.7.7 and Cor.16.6), an orientation of $H^1(H_1^g, \mathbb{R})$ defines an orientation of $H^1(H_2^g, \mathbb{R})$ if we require $i_1^\# \oplus i_2^\#$ being orientation preserving.
By excision (*exc*) and Poincare-Lefschetz duality (*PLD*) we obtain ($V_1 := H_1^1$):

$$H^1(H_1^g, H_1^g - k', \mathbb{R}) \overset{exc}{\cong} H^1(V_1, \partial V_1, \mathbb{R}) \overset{PLD}{\cong} H_2(V_1, \mathbb{R}) = 0 \ , \ H^2(H_1^g, \mathbb{R}) \overset{PLD}{\cong} H_1(H_1^g, \partial H_1^g, \mathbb{R}) = 0$$

and

$$H^2(H_1^g, H_1^g - k', \mathbb{R}) \overset{exc}{\cong} H^2(V_1, \partial V_1, \mathbb{R}) \overset{PLD}{\cong} H_1(V_1, \mathbb{R}) \cong H_1(k, \mathbb{R}) = \mathbb{R} \ .$$

Therefore $H^1(H_1^g - k, \mathbb{R})$ is oriented by the short exact sequence induced by the long exact sequence of the pair $(H_1^g, H_1^g - k)$:

$$0 \to H^1(H_1^g, \mathbb{R}) \to H^1(H_1^g - k', \mathbb{R}) \to H^2(H_1^g, H_1^g - k', \mathbb{R}) \to 0 \ .$$

By fixing an orientation of SU(2) we orient the 2-sphere S_α^2 (cf. [HK98], p.486). Furthermore, given an orientation of k, the meridians of k are oriented by the "right hand rule".
Thus, orienting SU(2) and F^g and specifying which handlebody to call H_1^g, fixes an orientation of all manifolds defined above. Note that a specification of one handlebody together with an orientation of F^g orients the homology 3-sphere Σ.

It should be mentioned that for $k = \emptyset$ the orientations correspond to the orientation of $\widehat{R}_1$ and $\widehat{R}_2$ used to define the Casson invariant (cf. [Sav99], Ch.16.2).

3.4 The intersection number $s^\alpha(k \subset \Sigma)$

Observing that the considered representation spaces are (open) manifolds with suitable dimensions, we are able to define the following intersection number:

Definition 3.4.1. *Let $\widehat{R}_1'^{\alpha}(\beta)$, $\widehat{R}_2$ and $\widehat{R}(F^g)$ be given as oriented manifolds. By choosing an isotopy $\widehat{R}_1'^{\alpha}(\beta) \rightsquigarrow \widetilde{R}_1'^{\alpha}(\beta)$ such that $\widetilde{R}_1'^{\alpha}(\beta) \pitchfork \widehat{R}_2$ the intersection number*

$$s^{\alpha}(k \subset \Sigma) := (-1)^g \sum_{p \in \widetilde{R}_1'^{\alpha}(\beta) \pitchfork \widehat{R}_2 \subset \widehat{R}(F^g)} \varepsilon_p \ , \ \varepsilon_p = \pm 1$$

is defined.

An argument completely analogous to that of the Casson invariant (cf. [Sav99], p.153) shows that all signs ε_p change if the orientation of F^g is reversed, or the roles of H_1^g and H_2^g are switched (where both cases correspond to reversing the orientation of Σ). If we reverse the orientation of k the orientations of $H^1(H_1^g - k', \mathbb{R})$ and of all knot meridians are reversed. By Fox calculus we obtain

$$su(2) \ni 0 = \mathrm{d}1 = \mathrm{d}(X^{-1}X) = \mathrm{d}X^{-1} + X^{-1} \circ \mathrm{d}X \Leftrightarrow \mathrm{d}X^{-1} = -X^{-1} \circ \mathrm{d}X \ .$$

Since $\circ$ is orientation preserving the reversion of knot meridians changes the orientation of $T_p\widehat{R}_1'^{\alpha}(\beta)$ by a factor -1. This factor cancels with the factor -1 provided by reversing the orientation of $H^1(H_1^g - k')$.[1] Thus the intersection number $s^{\alpha}(k \subset \Sigma)$ is independent of the orientation of k and defined if we fix an orientation of Σ.

Because the intersected manifolds are open we further have to ensure the independence of the isotopies $\widehat{R}_1'^{\alpha}(\beta) \rightsquigarrow \widetilde{R}_1'^{\alpha}(\beta)$ and $\widehat{\Gamma}_{\beta}^{\alpha} \rightsquigarrow \widetilde{\Gamma}_{\beta}^{\alpha}$ of lemma 3.2.3. This holds if both can be chosen with compact support and, according to the next theorem, is always possible if the condition $\Delta_{k \subset \Sigma}(e^{2i\alpha}) \neq 0$ is satisfied. Note that our notation anticipates the fact that $s^{\alpha}(k \subset \Sigma)$ is actually a knot invariant (see Th.4.4.19). At the moment the definition depends on the plat presentation $\widehat{\beta}$ as well as the chosen Heegaard splitting (cf. Rem.3.2.4). In order to show that $s^{\alpha}(k \subset \Sigma)$ is independent of these choices we will examine the computation process of $s^{\alpha}(k \subset \Sigma)$.

Theorem 3.4.2. *If $\Delta_{k \subset \Sigma}(e^{2i\alpha}) \neq 0$ holds, a sufficiently small neighborhood of reducible representations of $\pi_1(\Sigma - k)$ consists entirely of reducible representations.*

Proof. Let $k' \subset H_1^g$ be given in standard position. A Mayer-Vietoris sequence for the decomposition $\Sigma = (\Sigma - N_k) \cup \overline{N}_k$, where N_k is an open tubular neighborhood of k, shows $H_1(\Sigma - k) = \mathbb{Z}$. Therefore all abelian representations of the knot complement factor through $H_1(\Sigma - k)$ which is generated by a meridian of k'. For k' in standard position the fundamental group $\pi_1(H_1^g - k')$ is generated by the longitudes l_i, $1 \leq i \leq g$, of H_1^g and the upper closing arcs x_{2i}, $1 \leq i \leq n$, of the $2n$-plat presentation $\widehat{\beta}$. For an abelian representation we obtain $L_i = 1 \in \mathrm{SU}(2)$, $1 \leq i \leq g$, and (modulo SO(3)-conjugation) $X_{2i} = (\alpha, \boldsymbol{e}_x)$, $1 \leq i \leq n$, for the representing matrices. Let R_i, $1 \leq i \leq n+g$, be the relations determining $\pi_1(\Sigma - k)$ according to the standard position of k' (see equation (3.3)). Since we use a homologically flat Heegaard diagram we infer from corollary 2.3.7 and the proof of theorem 3.2.5 that the first g relations determining the Zariski tangent space of $\pi_1(\Sigma - k)$ at the reducibles boil down to the abelianized derivatives $\mathrm{ab}(\partial R_i / \partial \gamma_j) = 0$, $1 \leq i \leq g$, $g+1 \leq j \leq n+g$, with l_i and x_{2i} substituted by $\mathrm{d}L_i$ and $\mathrm{d}X_{2i}$ respectively. This yields $\mathrm{d}L_i = 0$, $1 \leq i \leq g$, for the Zariski tangent vectors of the longitudes. Thus for the Zariski tangent vectors of the knot meridians at the reducibles of $\pi_1(\Sigma - k)$, the same conditions hold as for the Zariski tangent space at the reducibles of $\pi_1(S^3 - k')$. Then using theorem 2.2.8 the given statement is

[1] If $k \subset B^3 \subset S^3$ we have $g = 0$ and $H^1(B^3 - k, \mathbb{R}) \cong H^1(k, \mathbb{R})$. The factor -1 which results from changing the orientation of k cancels with an overall factor -1 of $T_p\widehat{\Gamma}_{\sigma}^{\alpha}$. Therefore $h^{\alpha}(k)$ defined in [HK98] is independent of the orientation of k.

proved by deducing a contradiction analogous to that in the proof of theorem 19 in [Kla91] . ■

Remark 3.4.3. Theorem 3.4.2 holds in the more general case of compact orientable 3-manifolds, which are rational homology circles with torus boundary (cf. [HPS01], Th.2.7,). We gave a proof for the special situation of a knot in a homology 3-sphere because some arguments will be used for the computation of $s^\alpha(k \subset \Sigma)$.

Theorem 3.4.2 leads to a criterion for the compactness of the intersection $\widehat{\imath}_1(\widehat{R}^\alpha(H_1^g - k')) \cap \widehat{R}_2$ in $\widehat{R}(F^g)$.

Corollary 3.4.4. *Let $k \subset \Sigma = H_1^g \cup H_2^g$ be a knot with $k \subset H_1^g$. If $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ holds then the intersection*

$$\widehat{\imath}_1(\widehat{R}_1^\alpha(H_1^g - k')) \cap \widehat{R}_2 \subset \widehat{R}(F^g)$$

is compact.

Since $\widehat{R}^\alpha(H_1^g - k') \subset \widehat{\Lambda}_{n,g}^\alpha$ it follows from equation (3.1) that $\widehat{\imath}_1 : \widehat{R}^\alpha(H_1^g - k') \to \widehat{R}(F^g)$ is a proper map. Therefore the preimage $\widehat{\imath}_1^{-1}(\widehat{R}^\alpha(H_1^g - k') \cap \widehat{R}_2) \subset \widehat{H}_{n,g}^\alpha$ is a compact subset and we can restrict the isotopy $\widehat{\Gamma}_\beta^\alpha \rightsquigarrow \widetilde{\Gamma}_\beta^\alpha$ to a compact subset of $\widehat{H}_{n,g}^\alpha$. As a consequence $\widehat{R}_1'^\alpha(\beta) = \widehat{\Lambda}_{n,g}^\alpha \pitchfork \widetilde{\Gamma}_\beta^\alpha$ is a compact subset of $\widehat{H}_{n,g}^\alpha$ as well. Therefore the intersection $\widehat{\imath}_1(\widehat{R}_1'^\alpha(\beta)) \cap \widehat{R}_2 \subset \widehat{R}(F^g)$ is compact too. We summarize the results in

Corollary 3.4.5. *If $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ the isotopies $\widehat{\Gamma}_\beta^\alpha \rightsquigarrow \widetilde{\Gamma}_\beta^\alpha$ in $\widehat{H}_{n,g}^\alpha$ and $\widehat{R}_1^{\alpha\prime}(\beta) \rightsquigarrow \widetilde{R}_1^{\alpha\prime}(\beta)$ in $\widehat{R}(F^g)$ can be chosen with compact supports.*

It follows

Corollary 3.4.6. *Let $k \subset \Sigma$ be a knot in a homology sphere. Then the intersection number*

$$s^\alpha(k \subset \Sigma) = (-1)^g \langle \widehat{R}_1^\alpha(\beta), \widehat{R}_2 \rangle_{\widehat{R}(F^g)}$$

is well defined for all α with $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$.

Remark 3.4.7. The homologically flat embedding which implies the conditions $L_i = 1$ and $\mathrm{d}L_i^1 = 0$, $1 \leq i \leq g$, still allows the representation space of $\Sigma - k$ at the reducibles to be regarded as if the knot were embedded in S^3 (where all conditions trivially hold).

A close look at $\alpha \in \{0, \pi\}$ reveals a situation quite opposite to that in 3.4.7. Then all meridians of k are represented by central matrices. Thus, from the representation spaces' point of view, k is not "visible". Due to the similarities of the constructions we expect

$$\lim_{\alpha\to 0,\pi} s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma) \ , \tag{3.4}$$

where the factor 2 corresponds to the additional factor 1/2 in the definition of $\lambda(\Sigma)$ (see Def.2.4.5). Because $s^\alpha(k \subset \Sigma)$ is not defined for the limits $\alpha \in \{0, \pi\}$ we are content with proving equation (3.4) for a knot $k_0 \subset \Sigma$ with trivial Alexander polynomial. This provides the starting point from which the general statement follows by the computation algorithm (see Cor.4.4.15).

Lemma 3.4.8. *Let $k_0 \subset \Sigma$ be a knot with trivial Alexander polynomial. Then*

$$\lim_{\alpha\to 0,\pi} s^\alpha(k_0 \subset \Sigma) = 2\lambda(\Sigma) \ .$$

Proof. We prove the statement for a knot $k_0 \subset \Sigma$ in standard position having one and only one maximum with respect to an axis perpendicular to the projective plane of $\widehat{\beta}$ (compare the proof of Cor.3.2.6). Furthermore we assume that maximum is at the *end* of the first handle l_1.

Let a collection of $L_i = \rho(l_i) \in \widetilde{R}_1 \pitchfork \widehat{R}_2$, $l_i \in \pi_1(H_1^g)$, $1 \leq i \leq g$, be given. We may assume $\{L_1, \ldots, L_n\}$ to be a non-abelian set (after a small isotopy if necessary). Choose $X_1 \in S_\alpha^2$ for the representation of the knot meridian x_1. Because of the monotony and since the L_i are fixed this determines the representations of all other meridians, especially $X_2 \in S_\alpha^2$. To obtain a representation of $\Sigma - k_0$ the equation $X_1 = L_1 \circ X_2$ must be satisfied. This is possible choosing a suitable $L_1 \in \mathrm{SU}(2)$.
The meridian x_2 follows from x_1 by conjugation with knot meridians and longitudes. If α approaches 0 or π (and since $1 \in R(\Sigma)$ is isolated) the conjugations by matrices representing the knot meridians are negligible compared with the conjugations by matrices representing the longitudes.

Let the order of the handles, for which the knot passes through, be given by $(i_1, \ldots, i_{n-1})$ where $\{i_1, \ldots, i_{n-1}\} = \{2, \ldots, n\}$. Then the spatial part of X_2 lies in a small neighborhood of

$$(L_{i_{n-1}}^{\varepsilon_{i_{n-1}}} \ldots L_{i_1}^{\varepsilon_{i_1}}) \circ X_1 \ , \ \varepsilon_{i_j} = \pm 1 \ .$$

The equation $X_2 = (L_{i_{n-1}}^{\varepsilon_{i_{n-1}}} \ldots L_{i_1}^{\varepsilon_{i_1}}) \circ X_1$ is satisfied for a X_1 having a spatial part parallel to the spatial part of $\prod_{j=1}^{n-1} L_{i_j}^{\varepsilon_{i_j}}$. Since $\{L_1, \ldots, L_n\}$ is a set of non-abelian matrices we can choose an isotopy $\widehat{R}_1'^{\alpha}(\beta) \rightsquigarrow \widetilde{R}_1'^{\alpha}(\beta)$ such that any $p \in \widetilde{R}_1 \pitchfork \widehat{R}_2$ corresponds with one and only one $p' \in \widetilde{R}_1'^{\alpha}(\beta) \pitchfork \widehat{R}_2$. From the orientation conventions and the definition of $s^\alpha(k \subset \Sigma)$ it follows that $\lim_{\alpha \to 0,\pi} s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma)$. ∎

In the following we want to show that the equation $s^\alpha(k_0 \subset \Sigma) = 2\lambda(\Sigma)$ holds for *all* $\alpha \in (0, \pi)$. For this purpose let us consider manifolds analogous to those defined in chapter 3.2 but all angles of the open interval $(0, \pi)$ admitted. Denoting these manifolds as before with α omitted we find

$$\widehat{\Lambda}_{n,g}^\alpha \subset \widehat{\Lambda}_{n,g} \ , \ \widehat{\Gamma}_\beta^\alpha \subset \widehat{\Gamma}_\beta \ , \ \widehat{H}_{n,g}^\alpha \subset \widehat{H}_{n,g} \ ,$$

being submanifolds of codimension one. The space of non-abelian representation classes of $\widehat{R}(H_1^g - k')$ can be identified with the intersection $\widehat{\Lambda}_{n,g} \cap \widehat{\Gamma}_\beta \subset \widehat{H}_{n,g}$ which is in general not compact. However, if the abelian representation ρ_α is not a limit of non-abelian representations the intersection $\widehat{i}_1(\widehat{R}(H_1^g - k')) \cap \widehat{R}_2 \subset \widehat{R}(F^g)$ is compact. From corollary 3.4.5 it follows that the intersection $\widehat{\Lambda}_{n,g}^\alpha \cap \widehat{\Gamma}_\beta^\alpha \subset \widehat{H}_{n,g}^\alpha$ is compact as well. Moreover, there exists an $\varepsilon > 0$ such that

$$(\widehat{\Lambda}_{n,g} \cap \widehat{\Gamma}_\beta) \cap \widehat{H}_{n,g}^{[\alpha-\varepsilon,\alpha+\varepsilon]} =: \widehat{H}_{n,g}^{\alpha,\varepsilon} \quad , \quad \widehat{H}_{n,g}^{[\alpha_1,\alpha_2]} := \bigcup_{\alpha \in [\alpha_1,\alpha_2]} \widehat{H}_{n,g}^\alpha$$

is compact (cf. [HK98], p.487). This is used in the following lemma.

Lemma 3.4.9. *Let k be a knot and in a homology 3-sphere Σ, and assume that the abelian representation ρ_α is not a limit of non-abelian representations. Then there exists an $\varepsilon > 0$ such that $s^\alpha(k \subset \Sigma) = s^\gamma(k \subset \Sigma)$ for $|\alpha - \gamma| < \varepsilon$.*

Proof. First we choose an $\varepsilon > 0$ such that $(\widehat{\Lambda}_{n,g} \cap \widehat{\Gamma}_\beta) \subset \widehat{H}_{n,g}^{\alpha,\varepsilon}$ is compact. Let $\widehat{\Gamma}_\beta^\alpha \rightsquigarrow \widetilde{\Gamma}_\beta^\alpha$ be an isotopy with compact support such that $\widetilde{\Gamma}_\beta^\alpha \pitchfork \widehat{\Lambda}_{n,g}^\alpha$. Extend this perturbation to an isotopy $\widehat{\Gamma}_\beta \rightsquigarrow \widetilde{\Gamma}_\beta$ with compact support such that $\widetilde{\Gamma}_\beta \pitchfork_{\widehat{H}_{n,g}^{\alpha,\varepsilon}} \widehat{\Lambda}_{n,g}$. Using the map $\widehat{i}_1 : \widehat{\Lambda}_{n,g} \to R(F^g)$ induced by inclusion (see equation (3.1)), we define

$$\widehat{R}_1' := \widehat{R}_1'(\beta) := \widehat{i}_1(\widetilde{\Gamma}_\beta \cap \widehat{\Lambda}_{n,g}) \ .$$

with $\widehat{R}_1'^{\alpha} := \widehat{R}_1'^{\alpha}(\beta) = \widehat{i}_1((\widetilde{\Gamma}_\beta \cap \widehat{\Lambda}_{n,g}) \cap \widehat{H}^{\alpha}_{n,g}) \subset \widehat{R}_1'$. Therefore the intersection

$$\widehat{R}_1'^{\alpha,\varepsilon} \cap \widehat{R}_2 \subset \widehat{R}(F^g)$$

is compact. Let $\widehat{R}_1'^{\alpha} \rightsquigarrow \widetilde{R}_1'^{\alpha}$ be an isotopy with compact support such that $\widetilde{R}_1'^{\alpha} \pitchfork \widehat{R}_2$. Extend this perturbation to an isotopy $\widehat{R}_1' \rightsquigarrow \widetilde{R}_1'$ with compact support such that $(\widehat{R}_1'^{[\alpha,\varepsilon]} \cap \widetilde{R}_1) \pitchfork \widehat{R}_2$. If $|\alpha - \gamma| < \varepsilon$ holds $\widetilde{R}_1'^{\gamma} := \widehat{R}_1'^{\gamma} \cap \widetilde{R}_1'$ is a $(3g-3)$-dimensional manifold with $\widetilde{R}_1'^{\gamma} \pitchfork \widehat{R}_2 \subset \widehat{R}(F^g)$. Then $\widetilde{R}_1' \cap \widehat{R}_2$ yields a 1-dimensional bordism from $\widetilde{R}_1'^{\alpha} \pitchfork \widehat{R}_2$ to $\widetilde{R}_1'^{\gamma} \pitchfork \widehat{R}_2$ and the statement follows. ■

From the lemmata 3.4.8 and 3.4.9 and because the interval $[0, \pi]$ is compact we obtain

Corollary 3.4.10. *Let $k_0 \subset \Sigma$ be a knot with trivial Alexander polynomial. Then $s^{\alpha}(k_0) = 2\lambda(\Sigma)$ holds for all $\alpha \in (0, \pi)$.*

If $\Sigma = S^3$ we obtain $s^{\alpha}(k \subset \Sigma) = h^{\alpha}(k)$ where $h^{\alpha}(k)$ is the invariant defined in [HK98], p.486. Analogously to the intersection number $h^{\alpha}(k)$ in [HK98] we interpret $s^{\alpha}(k \subset \Sigma)$ as counting the regular representations of the knot complement at a fixed angle α with signs. The given interpretation of $s^{\alpha}(k \subset \Sigma)$ will be underlined by its computation using a skein algorithm which is done in the next chapter. The computation will also show that $s^{\alpha}(k \subset \Sigma)$ is actually a knot invariant.

Chapter 4

The computation of $s^\alpha(k \subset \Sigma)$

To compute $s^\alpha(k \subset \Sigma)$ we will determine the difference which comes up if we change a crossing of k. Therefore we consider

$$\Delta s^\alpha(k \subset \Sigma) = s^\alpha(k_+ \subset \Sigma) - s^\alpha(k \subset \Sigma) ,$$

where k_+ denotes the knot k with a crossing changed inside a 3-ball $B^3 \subset \Sigma$. In fact this is equivalent to changing the corresponding crossings of $k' \subset S^3$ (compare Th.2.2.8).

Let us first establish the *computational position* $k'_c \subset H_1^{g+1}$ for k' which means to "isolate" the crossing which is to be changed in an additional handle of H_1^g. Then the crossing is switched by performing a simple Dehn twist along the meridian of this handle. This induces an isotopy in the representation space which can be controlled. For this purpose we project the (in general high dimensional) manifolds used to define $s^\alpha(k \subset \Sigma)$ onto curves on the 2-dimensional pillow-case representing the longitude and the meridian of the additional handle.

4.1 The definition of the computational position $k'_c \subset H_1^{g+1}$

Let $k' \subset H_1^g$ in standard position be given by the $2n$-plat presentation $\widehat{\beta}$. Then we obtain the computational position by the following manipulations:

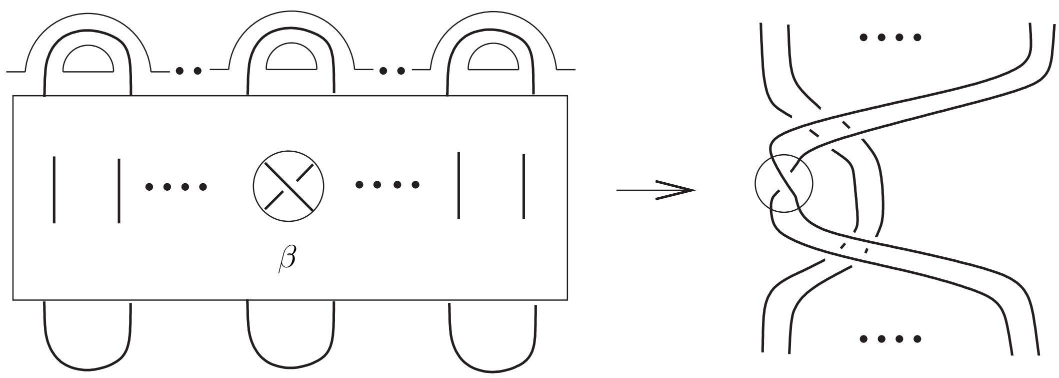


Figure 4.1: Establishing the computational position: Step 1.

1. Choose an isotopy such that the crossing which is supposed to be changed is presented by the generator $\sigma_1 \in \mathcal{B}_{2n}$ (see Fig.4.1).

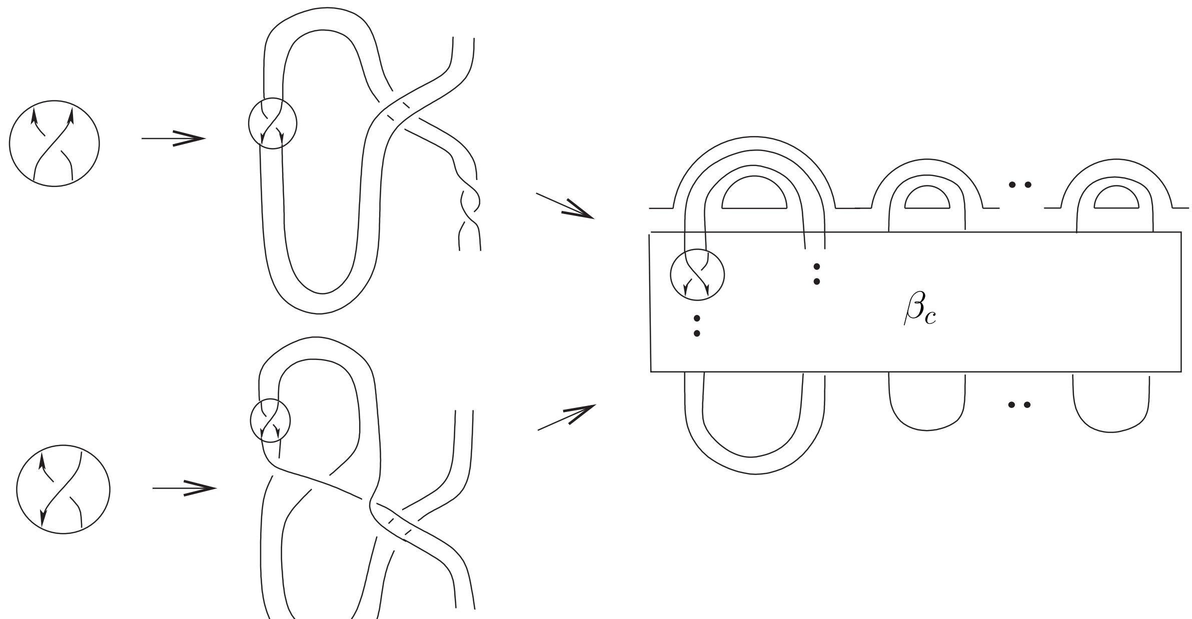


Figure 4.2: Establishing the computational position: Step 2.

2. Stabilize the new plat $\widehat{\beta}'$ and drill H_1^g as shown in figure 4.2 to obtain the knot in computational position $k_c' \subset H_1^{g+1}$. Since we want to achieve that both strands of k_c' pass the new handle in the same direction it depends on the crossing which of both stabilizations is needed. In each case the corresponding $(2n+2)$-braid is denoted by β_c.

Definition 4.1.1. *Let $k \subset \Sigma = H_1^g \cup H_2^g$ with $k' \subset H_1^g$ in standard position be given by the $2n$-plat $\widehat{\beta}$. Then we call the knot in* computational position *if the manipulations described above were carried out. The knot corresponding to the computational Heegaard diagram (H_1^{g+1}, h_c) is denoted by $k_c' \subset H_1^{g+1} \subset S^3$. It is presented by the $(2n+2)$-plat $\widehat{\beta}_c$.*

Note that the computational position, i.e. k_c', specifies the crossing which is to be changed and therefore the knot k_+.

By a construction analogous to the one in the definition of $s^\alpha(k \subset \Sigma)$ (cf. Sec.3.4) we define an intersection number $s^\alpha(k_c \subset \Sigma)$ for the knot in computational position.

Definition 4.1.2. *Let $\widehat{R}_1'^\alpha(\beta_c)$, $\widehat{R}_2' := \widehat{R}(H_1^{g+1})$ and $\widehat{R}(F^{g+1})$ be given as oriented manifolds. By choosing an isotopy $\widehat{R}_1'^\alpha(\beta_c) \rightsquigarrow \widetilde{R}_1'^\alpha(\beta_c)$ such that $\widetilde{R}_1'^\alpha(\beta_c) \pitchfork \widehat{R}_2'$ we define the intersection number*

$$s^\alpha(k_c \subset \Sigma) := (-1)^{g'} \sum_{p \in \widetilde{R}_1'^\alpha(\beta_c) \pitchfork \widehat{R}_2' \subset \widehat{R}(F^{g+1})} \varepsilon_p' \ , \ \ \varepsilon_p' = \pm 1 \ .$$

Since $s^\alpha(k \subset \Sigma)$ is not yet established as a knot invariant the definition is not redundant. In the following we show that the intersection numbers $s^\alpha(k \subset \Sigma)$ and $s^\alpha(k_c \subset \Sigma)$ are equal.

Let l_0 and m_0 denote the additional generators of $\pi_1(H_1^{g+1} - k_c')$ where $m_0 = (x_{-1}x_0)^{\varepsilon_0}$, $\varepsilon_0 = \pm 1$. Then the gluing homeomorphism is trivial with respect to the additional longitude of H_1^{g+1}, i.e. $h_{c*}(\partial D_0) = l_0$, which yields $l_0 = 1 \in \pi_1(\Sigma)$. Thus, according to [Sav99], p.154f, we obtain :

$$\widehat{R}_1'^\alpha(\beta_c) \cap \widehat{R}_2' = 1 \times \widehat{X_{-1}X_0}^{\varepsilon_0} \times \widehat{R}_1'^\alpha(\beta) \cap \widehat{R}_2 \ , \qquad (4.1)$$

where $\widehat{X_{-1}X_0} := \{X_{-1}X_0 \in \widehat{R}_1'^{\alpha}(\beta) \cap \widehat{R}_2\}$ and $X_{-1}X_0$ denotes the representation of a path enclosing the strands of the crossing which is to be changed.

To compute the intersection number $s^{\alpha}(k_c \subset \Sigma)$ the manifold $\widehat{R}_1'^{\alpha}(\beta_c)$ may need to be perturbed so that the intersection $\widetilde{R}_1'^{\alpha}(\beta_c) \cap \widehat{R}_2'$ is transversal. As in the case of the Casson invariant (see [AM90], p.77f) the perturbations can be chosen such that

$$\widetilde{R}_1'^{\alpha}(\beta_c) \pitchfork \widehat{R}_2' = 1 \times \widetilde{X_{-1}X_0}^{\varepsilon_0} \times (\widetilde{R}_1'^{\alpha}(\beta) \pitchfork \widehat{R}_2) \ , \ \widetilde{X_{-1}X_0} := \{X_{-1}X_0 \in \widetilde{R}_1'^{\alpha}(\beta) \pitchfork \widehat{R}_2\} \ . \tag{4.2}$$

As $s^{\alpha}(k_c \subset \Sigma)$ receives a "new" sign for the knot in computational position, namely $(-1)^{g+1} = -(-1)^g$, referring to the genus of the computational Heegaard diagram, we have to check that $\varepsilon_p' = -\varepsilon_p$ holds for all intersection points $p \in \widetilde{R}_1'^{\alpha}(\beta_c) \pitchfork \widehat{R}_2' \cong \widetilde{R}_1'^{\alpha}(\beta) \pitchfork \widehat{R}_2$.
To see this note that we need a permutation of the 1-dimensional vector space $\mathbb{R}$ with the g-dimensional vector space $H^1(H_1^g, \mathbb{R})$ to obtain the product orientation $H^1(H_2^{g+1}, \mathbb{R}) = \mathbb{R} \oplus H^1(H_2^g, \mathbb{R})$. Identifying $D_{\mathbf{X}}(X_{-1}X_0)^{\varepsilon_0} \in T_{(X_{-1}X_0)^{\varepsilon_0}}\mathrm{SU}(2)$ with $\mathrm{d}(X_{-1}X_0)^{\varepsilon_0} \in su(2)$ (see Th.2.3.6) we obtain

$$\begin{aligned} \varepsilon_p' T_p \widehat{R}(F^{g+1}) &= T_p \widehat{R}_1'^{\alpha}(\beta_c) \oplus T_p \widehat{R}_2' = (-1)^g su(2) \oplus T_p \widehat{R}_1'^{\alpha}(\beta) \oplus su(2) \oplus T_p \widehat{R}_2 \\ &= (-1)^g (-1)^{g-1} su(2) \oplus su(2) \oplus T_p \widehat{R}_1'^{\alpha}(\beta) \oplus T_p \widehat{R}_2 \\ &= -\varepsilon_p \, su(2) \oplus su(2) \oplus T_p \widehat{R}(F^g) = -\varepsilon_p T_p \widehat{R}(F^{g+1}) \end{aligned}$$

which proves the following lemma.

Lemma 4.1.3. *Let $k \subset \Sigma = H_1^g \cup H_2^g$ with $k' \subset H_1^g$ in standard position and let $k_c' \subset H_1^{g+1}$ be the corresponding knot in computational position. Then $s^{\alpha}(k \subset \Sigma) = s^{\alpha}(k_c \subset \Sigma)$ holds.*

4.2 An important example: the computation of $s^{\alpha}(k_n \subset S^3)$

It is useful to explain the procedure in the only non-trivial case with a Heegaard splitting of genus 1: the $(2,n)$-torus knots $k_n := \widehat{\sigma_1^n} \subset S^3$, $\sigma_1 \in \mathcal{B}_2$, with the 3-sphere decomposed into two solid tori V_i. In this case the knot k_n is already in computational position. Let the pillow case $\mathcal{PC}$ represent the non-central part of the commuting longitude l and meridian m of the common boundary $T^2 = \partial V_i$. Then any non-trivial representation of $\pi_1(S^3 - k_n)$ corresponds via $R(T^2) \ni M = X_{-1}X_0$ to a *non-central* element of the pillow-case. It should be mentioned that the representation spaces of torus knots are well known (cf. [Kla91], Th.1).

4.2.1 The case $k_n = k_1$

First we want to discuss the easiest case $k_0 \sim k_1 = \widehat{\sigma_1}$. This approach turns out to be important since the procedure can be generalized to representation spaces of arbitrary $(2,n)$-torus knots. But complications will arise from the more difficult relations in the fundamental group $\pi_1(S^3 - k_n)$ (if the latter is calculated from the braid representation of k_n).

Heegaard splitting of S^3 and orientations of the generators of $\pi_1(V_i)$

Let S^3 be decomposed into an "inner" torus V_2 and an "outer" torus V_1 with $k_1 \subset V_1$. For the fundamental groups we have

$$\pi_1(V_1) = \langle l_1 | - \rangle \approx \mathbb{Z} \ , \ \pi_1(V_2) = \langle l_2 | - \rangle \approx \mathbb{Z} \ , \ \pi_1(T^2) = \langle l, m | [m, l] \rangle \approx \mathbb{Z} \oplus \mathbb{Z} \ .$$

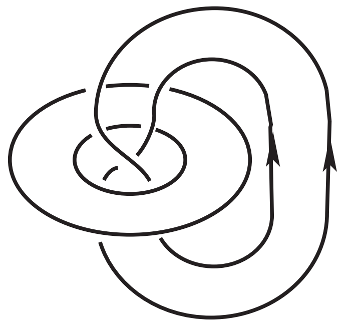


Figure 4.3: Decomposition of S^3 into an "inner" torus V_2 and an "outer" torus V_1 with $k_1 \subset V_1$.

where $T^2 = \partial V_i$ is the common boundary of the solid tori. The orientation of the boundary torus T^2 is chosen such that $\langle m, l, n\rangle$ is a positive oriented triple of vectors. Here l and m denote the vectors tangent to the canonical longitude and meridian respectively and n denotes an outward pointing normal vector n (see [Sav99], Fig.2.2).
Turning the inner torus V_2 inside out and reversing its orientation (which corresponds to the orientation conventions given in chapter 3.3) we identify the longitudes and meridians of the common boundary as

$$l_1 \leftrightarrow l \leftrightarrow m_2^{-1} \;\; , \;\; m_1 \leftrightarrow m \leftrightarrow l_2 \; .$$

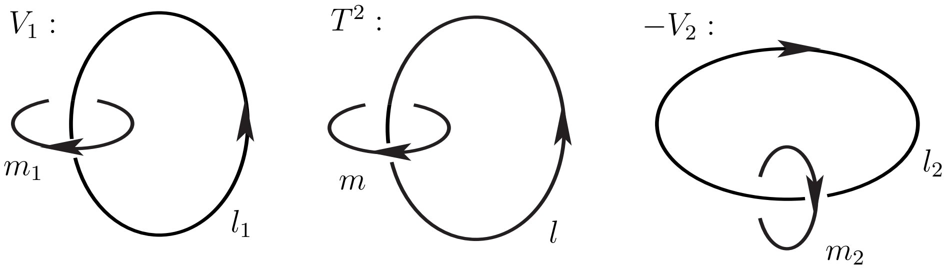


Figure 4.4: Orientations of the generators of $\pi_1(\partial V_1)$, $\pi_1(\partial V)$ and $\pi_1(-\partial V_2)$ respectively.

Thus the gluing homeomorphism $h : T^2 \to T^2$ preserves the orientation[1] and the automorphism of $\pi_1(T^2)$, corresponding uniquely to h (cf. [Rol76], Ch.2, Th.D4), is given by the matrix $h = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$. Orienting the generators of the fundamental group $\pi_1(V_1 - k_1)$ as shown in figure 4.4 we get the following relations for the meridians

$$x_1' = x_1 \circ x_2 \;\; , \;\; x_2' = x_1 = x_1 \circ x_1$$

which can be combined to yield $(x_1, x_2) = x_1 \circ (x_2, x_1)$. The x_i' result from $x_i' = l_1 \circ x_i$, $i = 1, 2$, and we obtain the equations

$$l_1 \circ (x_1, x_2) = x_1 \circ (x_2, x_1) \tag{4.3}$$

for the fundamental group $\pi_1(V_1 - k_1)$. For the fundamental group of the inner torus with reversed orientation we have $\pi_1(-V_2) = \langle l_2 | - \rangle$. Together with the relations

$$l_1 = l = m_2^{-1} = 1 \; , \; m_1 = m = l_2 = 1 \; ,$$

[1]In difference to [Sav99], p.22, where a homeomorphism reversing the orientation is used.

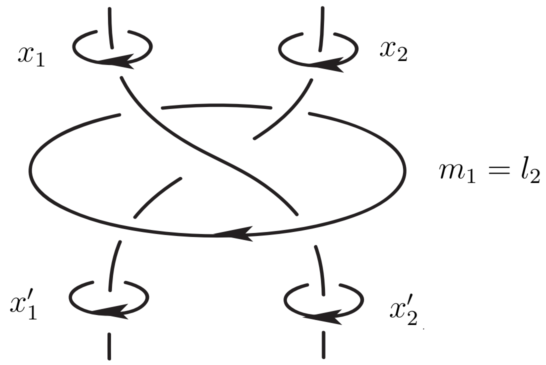


Figure 4.5: Orientation of the generators of $\pi_1(V_1 - k_1)$.

which are introduced by an ∞-surgery along the core of V_2, this leads to following presentation of the fundamental group of the trivial knot $k_1 \sim k_0 \subset S^3$:

$$\pi_1(V_1 - k_1 \cup_{h_\infty} -V_2) = \langle x_1, x_2 | x_1 = x_1 \circ x_2, x_1 = x_2 \rangle = \langle x_1 | - \rangle = \mathbb{Z} = \pi_1(k_0) ,$$

where $\pi_1(k_n) := \pi_1(S^3 - k_n)$. On the other hand, a +1-surgery yields relations of the form

$$l_1 m_1 = m_2^{-1} = 1 \Rightarrow l_1^{-1} = m_1 = x_1 x_2$$

and we obtain

$$\pi_1(V_1 - k_1 \cup_{h_{+1}} -V_2) = \langle x_1, x_2 | (x_1, x_2) = (l_1^{-1} x_1) \circ (x_2, x_1) = (x_1 x_2 x_1) \circ (x_2, x_1) \rangle = \pi_1(k_3) .$$

This is certainly not surprising because the +1-surgery corresponds to an additional 2π-left twist which leads to the (left handed) trefoil when regarding an embedded situation (see Fig.4.6).

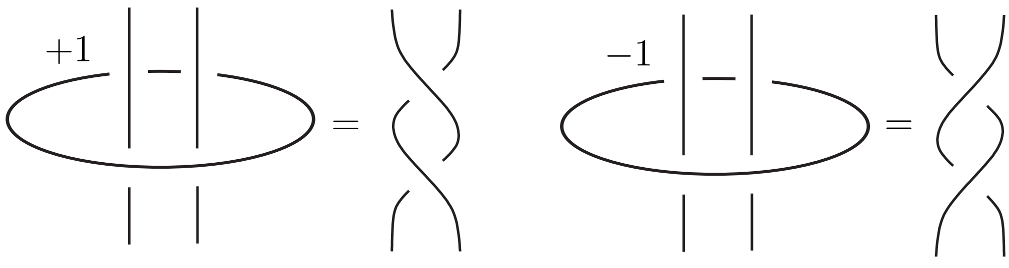


Figure 4.6: +1 and −1 correspond to additional 2π-left and 2π-right twists respectively.

The representation curves of the tori

The surjective maps $i_{j*} : \pi_1(T^2) \to \pi_1(V_j)$, $j = 1, 2$, induced by inclusions (compare Ch.2.4) lead to the embeddings

$$\begin{array}{rcl} \widehat{i}_j : \widehat{R}(V_j) & \to & \mathcal{PC} \\ [\rho] & \mapsto & [\rho \circ i_{j*}] \end{array}$$

of manifolds. Here $[\rho] := \rho/\mathrm{SO}(3)$ denotes the equivalence class of the representation ρ under the SO(3)-action. Recall that $\pi_1(T^2)$ is an abelian group and the pillow case represents the non-central part of its representation space. For the surgeries considered above we obtain the following results.

∞-surgery along the core of V_2: The equivalence classes of non the central representations R^{nc} of the fundamental groups $\pi_1(V_j)$ are homeomorphic to an open interval:

$$\widehat{R}(V_1) = \{\rho(l_1), \rho \in R^{nc}(V_1)\}/\mathrm{SO}(3) \cong (0,\pi) \cong \{\rho(l_2), \rho \in R^{nc}(-V_2)\}/\mathrm{SO}(3) = \widehat{R}(-V_2) \ .$$

The surjections

$$\begin{array}{rccc} i_{1*}: & \pi_1(T^2) & \to & \pi_1(V_1) \\ & l & \mapsto & l_1 \\ & m & \mapsto & m_1 = 1 \end{array} \quad , \quad \begin{array}{rccc} i_{2*}: & \pi_1(T^2) & \to & \pi_1(-V_2) \\ & l & \mapsto & m_2^{-1} = 1 \\ & m & \mapsto & l_2 \end{array} \tag{4.4}$$

lead to the following embedded curves

$$\begin{aligned} \widehat{i}_1(\widehat{R}(V_1)) &= \{\rho \circ i_{1*} \times \rho \circ i_{1*}(l,m), \rho \in R^{nc}(V_1)\}/\mathrm{SO}(3) \\ &= \{(\rho(l_1), \rho(m_1)), \rho \in R^{nc}(V_1)\}/\mathrm{SO}(3) = (0,\pi) \times \{0\} \subset \mathcal{PC} \ , \\ \widehat{i}_2(\widehat{R}(-V_2)) &= \{\rho \circ i_{2*} \times \rho \circ i_{2*}(l,m), \rho \in R^{nc}(-V_2)\}/\mathrm{SO}(3) \\ &= \{(\rho(m_2^{-1}), \rho(l_2)), \rho \in R^{nc}(-V_2)\}/\mathrm{SO}(3) = \{0\} \times (0,\pi) \subset \mathcal{PC} \ . \end{aligned}$$

Note that the run of the curves does not depend on the choice of the canonical parameterization of the pillow case.

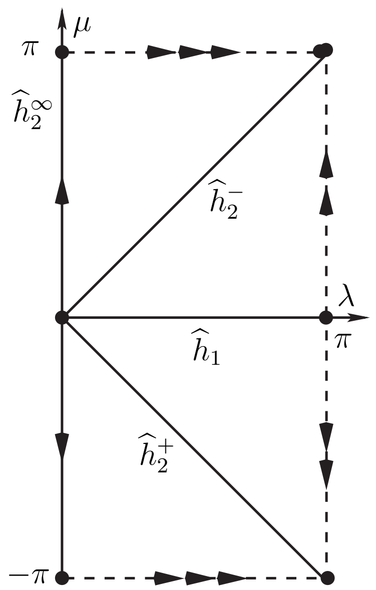


Figure 4.7: The representation curves of $\widehat{R}(V_1)$ and of $\widehat{R}(-V_2)$ for +1, −1 and ∞-surgery.

+1-surgery along the core of V_2: Denoting the embeddings with respect to a $\pm\frac{1}{n}$-surgery by $\widehat{i_j^{\pm\frac{1}{n}}}$ we have as in the case of ∞-surgery above $\widehat{i}_1^{+}(\widehat{R}(V_1)) = (0,\pi) \times \{0\} \subset \mathcal{PC}$. From the +1-surgery we obtain the relations

$$m_2^{-1} = lm \quad \text{and} \quad l_2 = m \tag{4.5}$$

which lead to

$$\rho(m_2^{-1}) = 1 = \rho(l)\rho(m) = \rho(l)\rho(l_2) \Rightarrow \rho(l) = \rho(l_2^{-1}) \quad \text{and} \quad \rho(m) = \rho(l_2) \;\; \text{respectively} \;\; . \qquad (4.6)$$

Using equation (2.8) and regarding the chosen parameterization of the pillow case (cf. Rem.2.3.5) it follows

$$\widehat{i_2^+}(\widehat{R}(-V_2)) = \{(l_2, -l_2), 0 < l_2 < \pi\} \; .$$

Similarly we obtain for arbitrary $\pm\frac{1}{n}$-surgeries along the core of V_2:

Lemma 4.2.1. *Let $S^3 = V_1 \cup_h V_2$ be a Heegaard splitting. If we perform a $\pm\frac{1}{n}$-surgery along the core of V_2, $n \in \mathbb{N}$, the embedding of $\widehat{R}(-V_2)$ is given by a curve with constant slope $\mp\frac{1}{n}$ on the pillow case.*

Proof. A $\pm\frac{1}{n}$-surgery along the core of V_2 induces (compare equation (4.5))

$$m_2^{-1} = lm^{\pm n} \quad \text{and} \quad l_2 = m \; .$$

Therefore we have to replace $\rho(l_2)$ by $\rho(l_2^{\pm n})$ in the first equation of (4.6) and obtain

$$\rho(l) = \rho(l_2^{\mp n}) \quad \text{and} \quad \rho(m) = \rho(l_2) \; .$$

Then equation (2.8) yields

$$\widehat{i_2^{\pm\frac{1}{n}}}(\widehat{R}(-V_2)) = \{(\pm n l_2, -l_2) \, , \, 0 < l_2 < \pi\}$$

for the curve embedded in the pillow case. ■

Remark 4.2.2. For the representations of the handlebodies which occur as embedded curves on the pillow case we will use the following notations:

$$\widehat{h}_1 := \widehat{i}_1(\widehat{R}(V_1)) \; , \; \widehat{h}_2 := \widehat{i}_2(\widehat{R}(-V_2)) \; , \; \widehat{h}_2^{\pm\frac{1}{n}} := \widehat{i_2^{\pm\frac{1}{n}}}(\widehat{R}(-V_2)) \; ,$$

with the abbreviations $\widehat{h}_2^{\pm 1} =: \widehat{h}_2^{\pm}$ and $\widehat{h}_2^{\infty} =: \widehat{h}_2$.

The representation curve of $\pi_1(V_1 - k_1)$ in the trace free case

The use of SU(2)-matrices with trace zero (i.e. having angle $\alpha = \pi/2$) provides an explanation of the procedure and at the same time minimizes the technical problems.
As a start we remark that having presented k_1 as a closed braid $k_1 = \sigma_1^{\wedge}$, the braid relation yields

$$x_1 x_2 = \sigma_1 \sigma_2 = x_1' x_2' = (x_1 \circ x_2) x_1 \; .$$

With the help of the relations (4.3) we have

$$l_1^{-1} \circ (x_1 x_2) = l_1^{-1} \circ ((x_1 \circ x_2) x_1) = l_1^{-1} \circ (x_1 \circ x_2) \; l_1^{-1} \circ x_1 = x_1 x_2 = m_1 \; , \qquad (4.7)$$

Therefore the spatial parts of the quaternions representing l_1 and m_1 respectively have to be parallel: $\boldsymbol{l_1} || \boldsymbol{x_1 x_2} || \boldsymbol{m}_1$. Here $\boldsymbol{x_1 x_2}$ denotes the spatial part of the product $X_1 X_2 \in \mathbb{H}_1$ (see Conv.2.5.1). This enables us to project the matrices L_1 and M_1 onto the pillow case. For the product of X_1 and X_2 equation (2.7) yields in the trace-free case:

$$X_1X_2 = \begin{pmatrix} c_\alpha \\ s_\alpha \boldsymbol{x_1} \end{pmatrix} \begin{pmatrix} c_\beta \\ s_\beta \boldsymbol{x_2} \end{pmatrix} \overset{\alpha=\beta=\pi/2}{=} \begin{pmatrix} -\boldsymbol{x_1} \cdot \boldsymbol{x_2} \\ \boldsymbol{x_1} \times \boldsymbol{x_2} \end{pmatrix} .$$

By SO(3)-conjugation we choose

$$X_1 = (\pi/2, c_\varphi \boldsymbol{e}_x + s_\varphi \boldsymbol{e}_y) \, , 0 \leq \varphi \leq \pi \quad \text{and} \quad X_2 = (\pi/2, \boldsymbol{e}_x) \, . \tag{4.8}$$

We determine L_1^{-1} in order to solve equation (4.7) where the solution is parameterized by the angle φ between the spatial parts of X_1 and X_2. Therefore the matrices have to satisfy

$$X_1 = L_1^{-1} \circ (X_1 \circ X_2) \ \ , \ \ X_2 = L_1^{-1} \circ X_1 \, . \tag{4.9}$$

Thus conjugation with $L_1^{-1} = \rho(l_1^{-1})$ must rotate the spatial parts of $X_1 \circ X_2$ and X_1 respectively by $-\varphi$ with axis $\boldsymbol{e}_z$. It seems trivial that one arrives at the same result with each of the following rotations:

1) by $-\varphi$ with axis $\boldsymbol{e}_z$, 2) by φ with axis $-\boldsymbol{e}_z$,

3) by $2\pi - \varphi$ with axis $\boldsymbol{e}_z$, 4) by $2\pi + \varphi$ with axis $-\boldsymbol{e}_z$,

but it is well worth having a close look at the quaternions behind. From equation (2.9) it can be seen that the equivalences between 1) and 2) on the one hand and 3) and 4) on the other hand are due to an identity of quaternions. To see the equivalence between both pairs, observe that as a consequence of the 2-fold covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ a conjugation with $\pm Q = \pm(\alpha, \boldsymbol{q}) \in \mathrm{SU}(2) = \mathbb{H}_1$ induces the same rotation (namely by $2\alpha = 2(\pi + \alpha)$, compare lemma 2.5.2 and equation (2.10)). So, to determine L_1 in (4.9), we may restrict ourselves to quaternions of the form $L_1^{-1} = (-\varphi/2, \boldsymbol{e}_z)$ or $L_1^{-1} = (\pi + \varphi/2, -\boldsymbol{e}_z)$. Applying equation (2.10) to the latter we obtain $L_1^{-1} = \pm(-\varphi/2, \boldsymbol{e}_z)$ which reflects the explicit form of the 2-fold covering: $\mathrm{SO}(3) = \mathrm{SU}(2)/\{\pm 1\}$.

Remark 4.2.3. In spite of the fact that the rotations of each pair are caused by identical quaternions, the choice of the spatial part (which distinguishes both cases) is not arbitrary. For the representation on the pillow case it is important to use those spatial parts to which the parameter angles λ and μ of the pillow case refer to. In our case both are related to $\boldsymbol{e}_z$.

Therefore, using the equations (2.9) and (2.10) again, we obtain

$$L = L_1 = \pm(\varphi/2, \boldsymbol{e}_z) = \begin{cases} c_{\varphi/2} + s_{\varphi/2}\boldsymbol{e}_z & = (\varphi/2, \boldsymbol{e}_z) \\ c_{\pi+\varphi/2} + s_{\pi+\varphi/2}\boldsymbol{e}_z & = (\pi + \varphi/2, \boldsymbol{e}_z) \end{cases} .$$

For the meridian of V_1 the parameterization by φ together with equation (2.8) yields

$$M = M_1 = X_1X_2 = \begin{pmatrix} -c_\varphi \\ -s_\varphi \boldsymbol{e}_z \end{pmatrix} = -(\varphi, \boldsymbol{e}_z) = (\pi + \varphi, \boldsymbol{e}_z), \ 0 \leq \varphi \leq \pi \, .$$

Regarding the chosen parameterization of the pillow case (see Rem.2.3.5) we obtain the following curve representing $\pi_1(V_1 - k_1)$:

$$\widehat{p}^{\pi/2}(\sigma_1) := \widehat{i}_1(\widehat{R}^{\pi/2}(V_1 - \widehat{\sigma_1})) = \{(-\pi + \varphi, \varphi/2), \varphi \in (0, 2\pi)\} \subset \mathcal{PC} \, .$$

We call $\widehat{p}^{\pi/2}(\sigma_1)$ the *representation curve* of σ_1. The intersection points of $\widehat{p}^{\pi/2}(\sigma_1)$ with $\widehat{h}_2$ and $\widehat{h}_2^+$ can be identified with irreducible representations of $\pi_1(k_1) = \pi_1(k_0)$ and $\pi_1(k_3)$ respectively. One case of reducible representations of $\pi_1(V_1 - k_1)$ is given by the representations $X_1 = X_2 = (0, \boldsymbol{e}_x)$ for the meridians x_i inducing $\mu = \arccos(1/2\mathrm{tr}\,(X_1X_2)) = \pi$. Moreover, the angle of the conjugating matrix $L_1 = (\lambda_1, \boldsymbol{e}_x)$ is arbitrary, i.e. $\lambda_1 \in (0, \pi)$. So we have a first interval $(0,\pi) \times \{\pi\} \sim (0,\pi) \times \{-\pi\} \subset \mathcal{PC}$ of reducible representations. The other possible case of abelian representation of $\pi_1(V_1 - k_1)$, namely $X_1 = X_2^{-1}$, only provides the point $(\pi/2, 0) \in \mathcal{PC}$ because $L_1 = -1$ is fixed in this situation. Hence all the reducible representations of $\pi_1(V_1 - k_1)$ are given by

$$\widehat{p}^{\pi/2}_{red}(\sigma_1) = (0,\pi) \times \{\pi\} \cup \{\pi/2\} \times \{0\} \subset \mathcal{PC} \,.$$

The reducible representations of $\pi_1(k_1)$ and $\pi_1(k_3)$ follow from the intersection with $\widehat{h}_2$ and $\widehat{h}_2^+$ respectively. Let us briefly summarize the results:

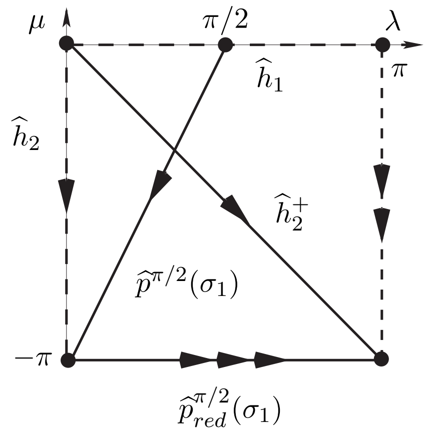


Figure 4.8: The representation curves $\widehat{p}^{\pi/2}(\sigma_1)$ and $\widehat{p}^{\pi/2}_{red}(\sigma_1)$ in the trace-free case.

Irreducible representations: The irreducible representations of $\pi_1(k_1)$ and $\pi_1(k_3)$ can be identified with $\widehat{p}^{\pi/2}(\sigma_1) \cap \widehat{h}_2 = \emptyset$ and $\widehat{p}^{\pi/2}(\sigma_1) \cap \widehat{h}_2^+ = (\pi/3, -\pi/3) \in \mathcal{PC}$ respectively.

Reducible representations: The reducible representations of $\pi_1(k_1)$ and $\pi_1(k_3)$ can both be identified with $\overline{\widehat{p}^{\pi/2}_{red}(\sigma_1)} \cap \overline{\widehat{h}_2} = \overline{\widehat{p}^{\pi/2}_{red}(\sigma_1)} \cap \overline{\widehat{h}_2^+} = (0,\pi) \in \overline{\mathcal{PC}}$, where for each space X we let $\overline{X}$ denote the closure of X. It should be mentioned that we need the closed sets only in the trace free case (cf. the next section). The point $(0, \pi)$ corresponds to the angle $\pi/2$ of the open interval $(0, \pi)$ which represents the non-central part of the reducibles of the knot complement.

Representation curve of k_1: The curve $\widehat{p}^{\pi/2}(\sigma_1)$ has slope 2. It consists of two branches each of which is related to one sheet of the 2-fold covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$. Both branches run from the reducible limits at (π, π) and $(0, -\pi)$ respectively, into $(\pi/2, 0) \in \mathcal{PC}$ and they are symmetric with respect to $(\pi/2, 0) \in \mathcal{PC}$. Hence we picture only the lower part of the pillow case with $-\pi \le \mu \le 0$ (see Fig.4.8).

Intersection number $s^\alpha(k)$: Choosing $T_p\mathcal{PC} = \{(\mathrm{d}\lambda, \mathrm{d}\mu), \mathrm{d}\lambda, \mathrm{d}\mu \in \mathbb{R}\}$ the orientations conventions of chapter 3.3 force the representation curves $\widehat{h}_i$ of the handlebodies V_i, $i = 1, 2$, to start at $(0,0) \in \overline{\mathcal{PC}}$. Similarly the orientation conventions let both branches of $\widehat{p}^{\pi/2}(\sigma_1)$ start at $(0,\pi/2) \in \mathcal{PC}$ and run into the reducible limits. Because $\widehat{p}^{\pi/2}(\sigma_1) \cap \widehat{h}_2 = \emptyset$ we obtain $s^\alpha(k_1 \subset S^3) = 0$. Evaluating $\langle \widehat{p}^{\pi/2}(\sigma_1), \widehat{h}_2^+ \rangle_{\mathcal{PC}}$ at $p = (\pi/3, -\pi/3) \in \mathcal{PC}$ yields

$$s^\alpha(k_1 \subset S^3) = (-1)^1 1 = -1 \tag{4.10}$$

for the *lefthanded* trefoil.

Casson invariant: By definition the Casson invariant of Σ is intersection number $-\langle \widehat{h}_1, \widehat{h}_2 \rangle_{\mathcal{PC}}$. Because $\widehat{h}_1 \cap \widehat{h}_2 = \emptyset$ we have $\lambda(S^3) = 0$ (as demanded in the definition of $\lambda(\Sigma)$).

The case $k = k_1 \subset S^3$ for arbitrary holonomy

According to theorem 4.2.5 the representation curve $\widehat{p}^\alpha(\sigma_1) := \widehat{i}_1(\widehat{R}^\alpha(V_1 - \widehat{\sigma_1}))$ is a curve with constant slope 2 on the pillow case for *all* $\alpha \in (0, \pi)$.

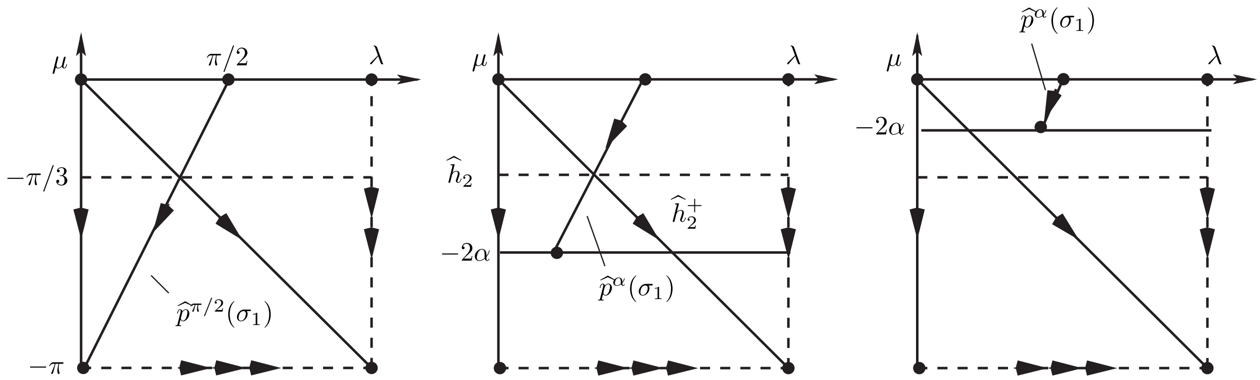


Figure 4.9: The representation curve $\widehat{p}^\alpha(\sigma_1)$ for different angles α.

Thus the main difference to the trace free case lies in the range of μ-values. To see this we note that the angle of the matrix representing m_1 is:

$$\arg(M_1) = \arg(X_1 X_2) = \arccos(c_\alpha^2 - s_\alpha^2 c_\varphi) \ .$$

For the irreducible representations with $c_\varphi \in (-1,1)$ we find the curve $\widehat{p}^\alpha(\sigma_1)$ being restricted to the area $[0,\pi] \times (-\arg(c_{2\alpha}), \arg(c_{2\alpha})) = [0,\pi] \times (-2\alpha, 2\alpha)$ of the pillow case. Similar to the trace-free case, intersection points of $\widehat{p}^\alpha(\sigma_1)$ with $\widehat{h}_2$ and $\widehat{h}_2^+$ can be identified with the irreducible representations of $\pi_1(k_1)$ and $\pi_1(k_3)$ respectively and again equation (4.9) is satisfied by an arbitrary $L_1 \in \mathrm{SU}(2)$. Therefore the reducibles of $\pi_1(V_1 - k_1)$ are now given by

$$\widehat{p}^\alpha_{red}(\sigma_1) = (0,\pi) \times \{-2\alpha\} \cup [0,\pi] \times \{2\alpha\} \subset \mathcal{PC} \ .$$

We obtain $\widehat{p}^\alpha_{red}(\sigma_1) \cap \widehat{h}_2 = (0, 2\alpha)$ for the reducibles of $\pi_1(k_0)$ and $\widehat{p}^\alpha_{red}(\sigma_1) \cap \widehat{h}_2^+ = (2\alpha, -2\alpha)$ for those of $\pi_1(k_3)$. Figure 4.9 shows the situation for different angles α.
As we will see (and as is known from [Kla91], Th.1) not all angles α will allow irreducible representations of the knot groups considered. More precisely: because $\widehat{p}^\alpha(\sigma_1) \cap \widehat{h}_2 = \emptyset$ for all $\alpha \in (0,\pi)$ we have no irreducible representations for $\pi_1(k_1)$ at all. But there are intersections of $\widehat{p}^\alpha(\sigma_1)$ and $\widehat{h}_2^+$ if $\mu < 0$ and

$$-\mu > \lambda \Rightarrow -\mu > \mu/2 + \pi/2 \Rightarrow \mu < \pi/3 \ .$$

Therefore we get irreducible representations of $\pi_1(k_3)$ for

$$\pi/6 < \alpha < 5\pi/6 \ .$$

The symmetry with respect to $\pi/2$ in this case follows trivially from $c_{2(\pi-\alpha)} = c_{2\alpha}$ but nevertheless reflects the symmetry of the representation spaces of knot groups in general. So the intersection number $s^\alpha(k_3)$, counting the intersection points with *positive* signs, changes at values α which via $t = e^{2i\alpha}$ are connected with the roots of the Alexander polynomial

$$\Delta_{k_3}(t) = t - 1 + t^{-1} = \frac{1}{t}(t - e^{i\pi/3})(t - e^{-i\pi/3}) \ .$$

As we know from theorem 3.4.2 the condition $\Delta_{k_3}(e^{2i\alpha}) = 0$ is necessary for $s^\alpha(k)$ to change. Later on it will be shown that comparing the signs of the Alexander polynomials of k and k_+ at $e^{2i\alpha}$ determines Δs^α (with $k = k_1$ and $k_+ = k_3$ in our example). This will turn out to be the key to the computation of $s^\alpha(k)$.

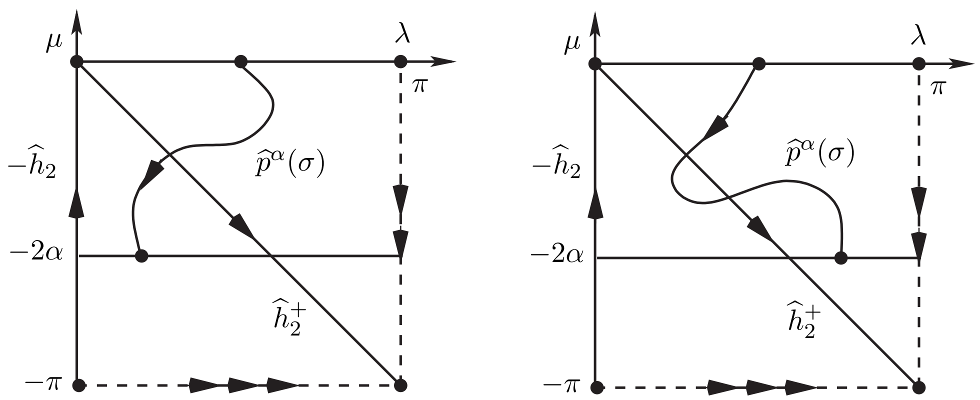


Figure 4.10: Situations relevant to $\Delta s^\alpha(k)$, $k = \widehat{\sigma}$.

We summarize the observations for arbitrary holonomy briefly:

Irreducible representations: Because $\widehat{p}^\alpha(\sigma_1) \cap \widehat{h}_2 = \emptyset$ for all $\alpha \in (0, \pi)$ there are no irreducible representations of $\pi_1(k_1) = \pi_1(k_0)$. The irreducible representations of $\pi_1(k_3)$ are given by

$$\widehat{p}^\alpha(\sigma_1) \cap \widehat{h}_2^+ = \begin{cases} (\pi/3, -\pi/3) \in \mathcal{PC} & \text{for } \alpha \in (\pi/6, 5\pi/6) \\ \emptyset & \text{for } \alpha < \pi/6 \ , \ \alpha > 5\pi/6 \end{cases} \ .$$

Reducible representation: The reducible representations of $\pi_1(k_1)$ and $\pi_1(k_3)$ can be identified with $\widehat{p}^\alpha_{red}(\sigma) \cap \widehat{h}_2 = (0, -2\alpha) \in \mathcal{PC}$ and $\widehat{p}^\alpha_{red}(\sigma_1) \cap \widehat{h}_2^+ = (2\alpha, -2\alpha) \in \mathcal{PC}$ respectively. Again the intersection points can be identified with the corresponding abelian and non-central representation of the knot complement, i.e. with $\alpha \in (0, \pi)$.

Intersection number $s^\alpha(k_1)$: If we change k_1 into k_3 the behavior of $s^\alpha(k_1)$ depends on the endpoint of $\widehat{p}^\alpha(\sigma_1)$ related to the oriented cycle

$$C_+ := \widehat{h}_2^+ \cup -\widehat{h}_2 \cup (0, \pi) \times \{\pi\} \ .$$

Because $s^\alpha(k_1)$ is an algebraic intersection number we deduce from figure 4.10

$$\Delta s^\alpha(k_1) = s^\alpha(k_3) - s^\alpha(k_1) \quad = \quad \begin{cases} 1, & \text{iff } \widehat{p}^\alpha(\sigma_1) \text{ ends in } C_+ \\ 0, & \text{iff } \widehat{p}^\alpha(\sigma_1) \text{ does not end in } C_+ \end{cases}$$

with

$$\Delta s^\alpha(k_1) \quad = \quad \begin{cases} 1 & \text{iff } \alpha \in (\pi/6, 5\pi/6) \\ 0 & \text{iff } \alpha < \pi/6 \ , \ \alpha > 5\pi/6 \end{cases}$$

in our example.
Let $\mu(\widehat{p}^\alpha(\sigma_1))$ and $\lambda(\widehat{p}^\alpha(\sigma_1))$ denote the ranges of μ and λ respectively. Because $\mu(\widehat{p}^\alpha(\sigma_1)) \subset (-2\alpha, 2\alpha)$, the intersection of $\widehat{p}^\alpha(\sigma_1)$ with $\widehat{h}_2^+ \cup -\widehat{h}_2$ already determines $\Delta s^\alpha(k_1)$. Thus we may write

$$\Delta s^\alpha(k_1) = \#(\widehat{p}^\alpha(\sigma_1) \pitchfork \widehat{h}_2^+) - \#(\widehat{p}^\alpha(\sigma_1) \pitchfork \widehat{h}_2) = \#(\widehat{p}^\alpha(\sigma_1) \pitchfork (\widehat{h}_2^+ - \widehat{h}_2)) \ ,$$

where $\widehat{p}^\alpha(\sigma_1)$ has to be transversal to both curves simultaneously.

The limiting behavior of $\widehat{p}^\alpha(\sigma_1)$: Due to the 2-fold covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ we have endpoints of $\widehat{p}^\alpha(\sigma_1)$ with $\lim_{\varphi\to 0} \mu(\widehat{p}^\alpha(\sigma_1)) = \mp 2\alpha$. The corresponding limits for $\lim_{\varphi \to 0} \lambda(\widehat{p}^\alpha(\sigma_1)) =: \frac{1}{2}\arg(\pm\lambda_0^\alpha)$ which are related to the conjugation with $\pm L_1^{-1}$ describe rotations in the tangent space at the reducible representation $\pi_1(V_1 - k_1)$ (compare Rem.2.3.4). By remark 2.5.4 the angels of these rotations evaluate to $\arg(\pm\lambda_0^\alpha)$ which yields identical rotations. The latter follows from remark 4.2.3 and is due to the symmetry with respect to $(\pi/2, 0) \in \mathcal{PC}$. Note that in our example the rotations in the tangent space at the reducibles of $\pi_1(V_1 - k_1)$ are trivial only in the trace-free case. In this case the endpoints of $\widehat{p}^{\pi/2}(\sigma_1)$ coincide with the cone points of $\mathcal{PC}$.

Remark 4.2.4. Note that the representation curves $\widehat{p}^\alpha(\sigma_1)$ for angles $\alpha < \pi/2$ and $\gamma := \pi - \alpha > \pi/2$ coincide. This can be seen as follows. Suppose the conjugation with L_0^{-1} induces a rotation by an angle φ for some $\alpha < \pi/2$ corresponding to the points $(\varphi, -2\alpha)$ and $(\pi - \varphi, 2\alpha)$ on the branches of the representation curve. Therefore we have to rotate by an angle of $2\pi - \varphi$ if the meridians are represented by matrices with trace $2\cos\gamma$. This corresponds to the λ-values $\pi - \varphi/2$ and $\varphi/2$ but at each case on the other branch. Thus the representation curve remains the same.

To complete the example we compute the representation curves and the intersection number s^α for arbitrary $(2, n)$-torus knots.

4.2.2 Computation of $s^\alpha(k_n \subset S^3)$

For the $(2, n)$-torus knot $k_n \subset S^3$, $n = 2l+1$, we use the same oriented generators of the fundamental group $\pi_1(V_1 - k_n)$ as shown in figure 4.5. Then the relations of $\pi_1(V_1 - k_n)$ induce the following equations of SU(2)-matrices representing the several generators:

$$L_1 \circ (X_1, X_2) = X(k_{2l+1}) \circ (X_2, X_1) \ \ , \ \ X(k_{2l+1}) = (X_1X_2)^l X_1 \ . \tag{4.11}$$

If P_{12} denotes the quaternion which permutes X_1 and X_2 the matrix

$$L_1 = X(k_n) P_{12}$$

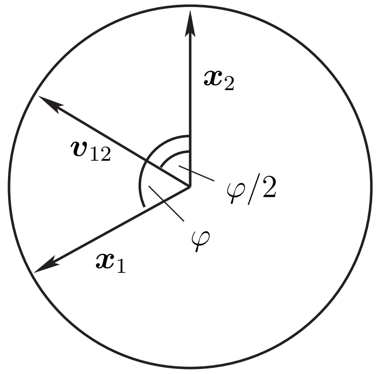


Figure 4.11: Spatial parts of the quaternions X_1, X_2 and V_{12}.

solves equation (4.11). As before, we choose $X_1 = c_\alpha c_\varphi + s_\alpha s_\varphi \boldsymbol{e}_x$ and $X_2 = c_\alpha + s_\alpha \boldsymbol{e}_x$ by SO(3)-conjugation and obtain (see Fig.4.11)

$$P_{12} = c_{\pi/2} + s_{\pi/2} \frac{\boldsymbol{x}_1 + \boldsymbol{x}_2}{|\boldsymbol{x}_1 + \boldsymbol{x}_2|} = \frac{1}{2c_{\varphi/2}}(\boldsymbol{x}_1 + \boldsymbol{x}_2) = c_{\varphi/2}\boldsymbol{e}_x + s_{\varphi/2}\boldsymbol{e}_y \; .$$

Writing the product of X_1 and X_2 as

$$X_1 X_2 = \begin{pmatrix} c_\alpha^2 - s_\alpha^2 c_\varphi \\ c_\alpha s_\alpha (1 + c_\varphi)\boldsymbol{e}_x + c_\alpha s_\alpha s_\varphi \boldsymbol{e}_y - s_\alpha^2 s_\varphi \boldsymbol{e}_z \end{pmatrix} =: \begin{pmatrix} c_\beta \\ s_\beta \boldsymbol{v} \end{pmatrix} , \tag{4.12}$$

we obtain

$$\arg M_1 = \arg(X_1 X_2) = \beta$$

for the matrix M_1 representing the meridian of V_1 and

$$\widehat{p}^\alpha(\sigma_1^n) := \widehat{i}_1(\widehat{R}^\alpha(V_1 - \widehat{\sigma_1^n})) = \{(\arg L_1(\beta), \beta),\ \beta \in (-2\alpha, 2\alpha)\} \subset \mathcal{PC} \; .$$

for the representation curve of σ_1^n. The curve is symmetric with respect to the point $(0, \pi/2) \in \mathcal{PC}$ related to the parameter value $\beta = 0 \Leftrightarrow \varphi = \pi$. The endpoints of $\widehat{p}^\alpha(\sigma_1^n)$ corresponding to the limits $\beta \to \pm\pi$ lie in the intervals $[0, \pi] \times \{\pm 2\alpha\} \subset \mathcal{PC}$. Considering the new parameter β we derive $\widehat{p}^\alpha(\sigma_1^n)$ from

$$\arg L_1(\beta) = \arccos((X_1 X_2)^l X_1 P_{12}) = \arccos((X_1 X_2)^{l+1} X_2^{-1} P_{12}) \; .$$

The last step will simplify the computations as follows. Let $U \in \mathrm{SU}(2)$ be the matrix which maps $X_1 X_2$ into the matrix $(\beta, \boldsymbol{e}_x)$ by conjugation, i.e.

$$U \circ (X_1 X_2) = c_\beta + s_\beta \boldsymbol{e}_x = \mathbf{e}^{i\beta} \in S^1 \subset \mathbb{H}_1 \; .$$

We obtain

$$(X_1 X_2)^{l+1} = U^{-1}(c_\beta + s_\beta \boldsymbol{e}_x)^{l+1} U = c_{(l+1)\beta} + s_{(l+1)\beta}\boldsymbol{v}$$

and therefore

$$(X_1 X_2)^{l+1} X_2^{-1} = \begin{pmatrix} c_{(l+1)\beta} \\ s_{(l+1)\beta}\boldsymbol{v} \end{pmatrix} \begin{pmatrix} c_\alpha \\ -s_\alpha \boldsymbol{e}_x \end{pmatrix} = \begin{pmatrix} c_{(l+1)\beta} c_\alpha + s_{(l+1)\beta} s_\alpha \boldsymbol{v} \cdot \boldsymbol{e}_x \\ c_\alpha s_{(l+1)\beta}\boldsymbol{v} - c_{(l+1)\beta} s_\alpha \boldsymbol{e}_x - s_{(l+1)\beta} s_\alpha \boldsymbol{v} \times \boldsymbol{e}_x \end{pmatrix} .$$

The relations $\boldsymbol{v}\cdot\boldsymbol{e}_x = \frac{1}{s_\beta} c_\alpha s_\alpha (1+c_\varphi)$ and $\boldsymbol{v}\times\boldsymbol{e}_x = -\frac{1}{s_\beta}(s_\alpha^2 s_\varphi \boldsymbol{e}_y + c_\alpha s_\alpha s_\varphi \boldsymbol{e}_z)$ as well as use of the addition theorems, the formulas for the half-angles and the relation $s_\alpha^2 c_\varphi = c_\alpha^2 - c_\beta$ yields:

$$(X_1X_2)^{l+1}X_2^{-1} = \begin{pmatrix} \frac{c_\alpha}{c_{\beta/2}} c_{(l+1/2)\beta} \\ s_\alpha \left[\left(\frac{c_\alpha^2}{s_\beta} s_{(l+1)\beta}(1+c_\varphi) - c_{(l+1)\beta} \right) \boldsymbol{e}_x + \frac{s_\varphi}{s_\beta} s_{(l+1)\beta} \boldsymbol{e}_y \right] \end{pmatrix} .$$

With the help of $s_\varphi = c_{\varphi/2}(1-c_\varphi)$ we compute

$$\cos(\arg L_1) = -(\boldsymbol{x}_1\boldsymbol{x}_2)^{l+1}\boldsymbol{x}_2^{-1}\cdot \boldsymbol{p}_{12} = -s_\alpha c_{\varphi/2}\left[\frac{s_{(l+1)\beta} c_{\beta/2}}{s_{\beta/2}} - c_{(l+1)\beta} \right] .$$

Using $c_\beta = c_\alpha^2 - s_\alpha^2 c_{\varphi/2}$, for the angles in question (namely $\alpha,\varphi\in[0,\pi]$) implies: $s_\alpha c_{\varphi/2} = s_{\beta/2}$. We conclude

$$-(\boldsymbol{x}_1\boldsymbol{x}_2)^{l+1}\boldsymbol{x}_2^{-1}\cdot \boldsymbol{p}_{12} = -s_{\beta/2}\left[\frac{c_{\beta/2}}{s_{\beta/2}} s_{(l+1)\beta} - c_{(l+1)\beta} \right] = -s_{(l+1/2)\beta} = \cos(\pi/2 + (l+1/2)\beta)$$

and finally

$$\arg L_1(\beta) = \pi/2 + \frac{2l+1}{2}\beta = \pi/2 + \frac{n}{2}\beta .$$

This implies

Theorem 4.2.5. *Let $k_n \subset S^3$ be a $(2,n)$-torus knot. Then the representation curve of k_n is given by*

$$\widehat{p}^\alpha(\sigma_1^n) = \left\{ (\frac{\pi}{2} + \frac{n}{2}\beta, \beta) \ , \ \beta \in (-2\alpha, 2\alpha) \right\} \subset \mathcal{PC} .$$

The theorem shows that the representation curves $\widehat{p}^\alpha(\sigma_1^n)$ have slope $\frac{\Delta\mu}{\Delta\lambda} = \frac{2}{n}$ for *all* $\alpha\in(0,\pi)$. Calculating their endpoints from the limit $\varphi\to 0$, theorem 4.2.5 yields

$$\lim_{\varphi\to 0} \widehat{p}^\alpha(\sigma_1^n) = (\frac{\pi}{2} \pm n\alpha, \pm 2\alpha) \subset \overline{\mathcal{PC}} \ , \ \alpha\in(0,\pi) .$$

Therefore $\widehat{p}^{\pi/2}(\sigma_1^n)$ has intersection points with $\widehat{h}_2$ at

$$2\alpha = \frac{2i+1}{n}\pi \Leftrightarrow \alpha = \frac{2i+1}{2n}\pi \ , \ 0 \le i \le \frac{n-3}{2} ,$$

and intersection points with $\widehat{h}_2^+$ at

$$2\alpha = \frac{2i+1}{n+2}\pi \Leftrightarrow \alpha = \frac{2i+1}{2n+4}\pi \ , \ 0 \le i \le \frac{n-1}{2} .$$

These points correlate via $t = e^{2i\alpha}$ with the roots of $\Delta_{k_n}(t)$ and $\Delta_{k_{n+2}}(t)$ respectively where ([Liv93], p.47)

$$\Delta_{k_n}(t) = \frac{1}{t^{\frac{n-1}{2}}} \frac{t^n+1}{t+1} .$$

Because the representation curves $\widehat{p}^\alpha(\sigma_1^n)$ are subsets of $\widehat{p}^{\pi/2}(\sigma_1^n)$ for $\alpha\neq\pi/2$ we obtain

Corollary 4.2.6. *Let $k_n \subset S^3$ and $\alpha\in(0,\pi)$ with $\Delta_{k_n}(e^{2i\alpha}) \neq 0$. Then $s^\alpha(k_n) = s^{\pi-\alpha}(k_n) = -\frac{[2n\alpha/\pi]-1}{2}$ where $[2n\alpha/\pi]$ denotes the smallest odd number greater than $2n\alpha/\pi$.*

Again the result reflects the symmetry of the representation spaces with respect to $\alpha = \pi/2$.

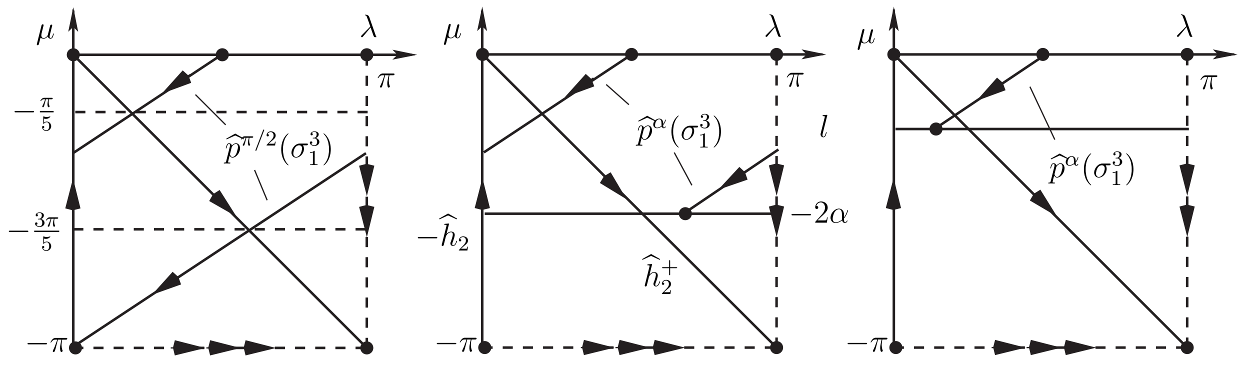


Figure 4.12: The representation curve $\widehat{p}^{\alpha}(\sigma_1^3)$ for different angles α.

Remark 4.2.7. It should be noted that it is possible to compute $s^{\alpha}(k_n)$ directly from the intersection of $\widehat{p}^{\alpha}(\sigma_1)$ and $\widehat{h}^{\frac{1}{n}}$. Explicitly this yields:

$$s^{\alpha}(k_n) = -\langle \widehat{p}^{\alpha}(\sigma_1), \widehat{h}^{\frac{1}{n}} \rangle_{\mathcal{PC}}$$

for the left-handed $(2,n)$-torus knots. Because $\widehat{h}^{-\frac{1}{n}}$ is a curve starting in the upper part of the pillow case (i.e. the part with $\lambda \in [0,\pi]$, see Lem.4.2.1) all signs of the intersection points in $\langle \widehat{p}^{\alpha}(\sigma_1), \widehat{h}^{-\frac{1}{n}} \rangle_{\mathcal{PC}}$ are reversed. Therefore we obtain

$$s^{\alpha}(k_n^*) = -s^{\alpha}(k_n)$$

for the right-handed $(2,n)$-torus knots k_n^*.

4.3 Comparison of $s^{\alpha}(k \subset \Sigma)$ and the Casson invariant $\lambda'(k)$

There are strong similarities between Casson's invariant for knots $\lambda'(k)$ and the intersection number $s^{\alpha}(k \subset \Sigma)$ which are sketched in the following. References for the statements related to the Casson invariant are [AM90] and [Sav99], Lecture 17.[2]

Roughly speaking, the computation of the Casson invariant $\lambda(\Sigma)$ is done by the surgery formula (2.6) which gives the changes of $\lambda(\Sigma)$ while performing a $\pm\frac{1}{n}$-surgery along a knot $k \subset \Sigma$ ([Sav99], Th.12.1). Together with the starting value $\lambda(S^3) = 0$ equation (2.6) determines $\lambda(\Sigma)$. This more formal definition of the Casson invariant does not involve the representation spaces. These come in by establishing the existence of $\lambda(\Sigma)$ and provide the interpretation as an algebraic intersection number (see Def.2.4.5). The sign $\pm 1 = \lambda'(k_{3_1})$ in (2.6) depends on the orientations of the manifolds intersected. As the notation indicates, the difference $\lambda'(k)$ is a knot invariant and independent from n.[3]

To prove the existence of $\lambda(\Sigma)$ a "preferred" Heegaard splitting $\Sigma = H_1^g \cup H_2^g$ is used. Within this splitting the knot k decomposes the boundary surface $F^g = F' \cup_k F''$ ([Sav99], Ch.17.1). A +1-surgery along k induces a diffeomorphism $\tau^* : \widehat{R}(F) \to \widehat{R}(F)$ which describes how the embeddings

[2] The arguments given in [Sav99], chapter 17.3, seem to be a little sketchy. We therefore refer to [AM90], chapter V.4, for that matter.

[3] It should be remarked that the independence from n is necessary to prove property P for $k \subset S^3$. For the computation of $\lambda(\Sigma)$ only the case $n = 0$, i.e. the difference $\lambda(\Sigma + k) - \lambda(\Sigma)$, is needed (cf. Rem.2.1.4).

of the representation spaces $\widehat{R}(H_i^g) =: \widehat{R}_i$, $i = 1, 2$, change. Using τ^* we can write

$$\lambda'(k) = -\frac{(-1)^g}{2} \# \left[\tau^* \widehat{R}_1 - \widehat{R}_1 \cap (\tau^*)^{n+1} \widehat{R}_2 \right] .$$

Due to the special position of k as a separating curve on F, the map τ^* can be realized by an isotopy in $\widehat{R}(F)$. Let δ' denote the cycle which is the boundary of this isotopy minus an open neighborhood of the reducibles of $R_1/\mathrm{SO}(3)$. Then a transversality argument shows that we obtain $\lambda'(k)$ as an intersection with the cycle δ'

$$\lambda'(k) = \#(\delta' \cap (\tau^*)^{n+1} \widehat{R}_2) .$$

Further let $\mathcal{N}$ denote a compact manifold neighborhood of $\widehat{R}_1 \cap \widehat{R}_-(F) \subset \widehat{R}_1$ with $\widehat{R}_-(F) = \{\rho : \pi_1(F) \to \mathrm{SU}(2) | \rho(k) = -1\}/\mathrm{SO}(3) \subset \widehat{R}(F)$. The completion of $\delta' -$ int $(\mathcal{N})$ along the trace of the isotopy yields a compact cycle β with

$$\#(\beta \cap (\tau^*)^{n+1} \widehat{R}_2) = 0 .$$

Thus the non trivial contribution to $\lambda'(k)$ is related to the *difference cycle* $\delta = \delta' - \beta$ and we have:

$$\lambda'(k) = \# \delta \cap (\tau^*)^{n+1} \widehat{R}_2 .$$

Now it is possible to collapse δ into a cycle δ_0 which entirely lies in $\widehat{R}_-(F)$. It follows:

$$\lambda'(k) = \# \delta \cap (\tau^*)^{n+1} \widehat{R}_2 \subset \widehat{R}_-(F) . \tag{4.13}$$

Because the action of τ^* on $\widehat{R}_-(F)$ is trivial the intersection number in (4.13) is independent of n and the knot invariant $\lambda'(k)$ is well defined. Moreover, the 2-fold covering $\mathrm{SU}(2) \to \mathrm{SO}(3)$ induces an even intersection number such that actually $\lambda(\Sigma) \in \mathbb{Z}$ ([AM90], Cor.4.8 and Ch.V.3 (a)). To complete the existence proof of $\lambda(\Sigma)$ the intersection number with the collapsed cycle $\delta_0 \subset \widehat{R}_-(F)$ is determined for the trefoil $k_{3_1} \subset S^3$ ([Sav99], Ch.17.5).
Now we want to show the analogies to the computation of $s^\alpha(k)$ within the realms of the discussed example $k_1 \subset S^3$. As stated before, the embeddings of the representation spaces of the tori V_i were given by the curves

$$\widehat{h}_1 = \widehat{i}_1(\widehat{R}(V_1)) = (0, \pi) \times \{0\} \ \ , \ \ \widehat{h}_2 = \widehat{i}_2(\widehat{R}(-V_2)) = \{0\} \times (0, \pi) .$$

After an application of a $+1$-surgery along V_2 we obtained the curve

$$\widehat{h}_2^+ = \widehat{i}^+(\widehat{R}(-V_2)) = \{(l_2, -l_2) \ , \ 0 \le l_2 \le \pi\} .$$

The transition from $\widehat{h}_2$ to $\widehat{h}_2^+$ can be realized by an isotopy on the pillow case. The construction relies on the fact that the exponential map

$$\begin{array}{rcccc} \exp & : & su(2) & \to & \mathrm{SU}(2) \\ & & A & \mapsto & e^A \end{array}$$

provides a diffeomorphism

$$\exp : B_\pi(0) \xrightarrow{\cong} \mathrm{SU}(2) \setminus \{-1\}$$

where $B_\pi(0)$ denotes the open 3-ball of radius π centered at the origin. As a consequence, the natural retraction $r : B_\pi(0) \times [0, 1] \to B_\pi(0)$ with $(A, t) \mapsto t \cdot A$ exponentiates to a retraction of $\mathrm{SU}(2) \setminus \{-1\}$:

$$r : \mathrm{SU}(2) \setminus \{-1\} \times [0, 1] \to \mathrm{SU}(2) \setminus \{-1\} \ , \quad (X, t) \mapsto X^t .$$

We construct an isotopy by means of this retraction:

$$H : R^{nc}(T^2) \times [0,1] \to R^{nc}(T^2)\ , \quad (L, M, t) \mapsto (L, M^{-t})\ .$$

Modulo SO(3)-conjugation this yields the desired isotopy $\widetilde{H}$ from $\widehat{h}_2$ to $\widehat{h}_2^+$ on the pillow case:

$$\begin{array}{rcccc} \widetilde{H} & : & \mathcal{PC} \times [0,1] & \to & \mathcal{PC} \\ & & (\lambda, \mu, t) & \mapsto & (\lambda, -t\mu) \end{array} .$$

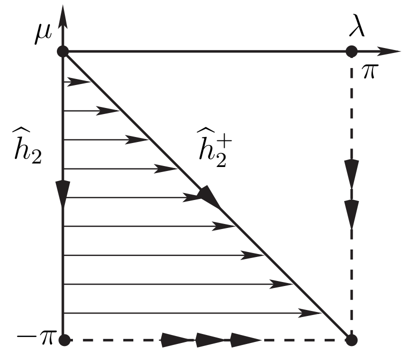


Figure 4.13: Realizing the Dehn surgery as an isotopy on the pillow case.

It follows that, similarly to the case of the Casson invariant, the effect of the Dehn surgery is determined by the boundary of an isotopy in the representation space of the boundary surface. But in opposition to the computation of $\lambda(\Sigma)$ the changes of $s^\alpha(k \subset \Sigma)$ can be completely controlled by the isotopy $\widetilde{H}$. In detail: if an endpoint of $\widehat{p}^\alpha(\sigma) \subset \mathcal{PC}$, $\widehat{\sigma} = k$, is met by $\widetilde{H}$ we obtain $\Delta s^\alpha(k \subset \Sigma) = 1$, otherwise $\Delta s^\alpha(k \subset \Sigma) = 0$.

This focuses the interest on the endpoints of the representation curve of the knot and therefore on the computation of the abelian limits of the non abelian representations. The computation will immediately yield that $s^\alpha(k \subset \Sigma)$ is a knot invariant. This marks a significant difference to the Casson invariant where the main interest is the existence proof.

Remark 4.3.1. For an arbitrary knot in a homology 3-sphere a crossing which is to be changed is isolated in an additional handle. For the pillow case representing longitude and meridian of this handle we have a similar isotopy (compare Ch.4.4).

4.4 Computation of s^α for arbitrary $k \subset \Sigma$

Let $k'_c \subset H_1^{g+1}$ be a knot in computational position. Then similarly to example $k_1 \subset S^3$ two strands of the knot go through one handle (compare Fig.4.5 and Fig.4.2). According to the equations (4.1) and (4.2) the elements of $\widehat{R}_1'^\alpha(\beta_c) \cap \widehat{R}_2'$ and $\widetilde{R}_1'^\alpha(\beta_c) \pitchfork \widehat{R}_2'$ satisfy the equation $\prod_{i=1}^g [L_i, M_i] = 1$. Furthermore in $\pi_1(F^{g+1})$ the relation $\prod_{i=0}^g [l_i, m_i] = 1$ holds. Thus we can restrict our considerations to submanifolds of $R(F^{g+1})$ whose matrices representing the additional longitude and meridian commute, i.e. $[L_0, M_0] = 1$. Setting $g' = g + 1$ we define a submanifold of $R(F^{g'})$ by

$$\begin{aligned} V_{g'} = \{ & (L_0, M_0, L_1, M_1, \ldots, L_g, M_g) \subset \mathrm{SU}(2)^{g'} | \\ & L_i, M_j \in h^*(R^{irr}(H_2^g)), 1 \le i, j \le g\ ,\ L_0, M_0 \in R^{nc}(T^2)\}\ , \end{aligned}$$

which respects $[L_0, M_0] = 1$. It is because of this relation for the dimension of $\widehat{V}_{g'}$ that

$$\dim \widehat{V}_{g'} = \dim(V_{g'}/\mathrm{SO}(3)) = 3g + 4 - 3 = 3g' - 2 \; .$$

Lemma 4.4.1. *Let $\widehat{V}_{g'}$ be defined as above. Then*

$$\langle \widehat{R}_1'^{\alpha}(\beta_c), \widehat{R}_2 \rangle_{\widehat{R}(F^{g'})} = \langle \widehat{V}_{g'} \pitchfork \widehat{R}_1'^{\alpha}(\beta_c), \widehat{R}_2' \rangle_{\widehat{V}_{g'}} \; .$$

Proof. Because

$$\dim \widehat{V}_{g'} + \dim \widetilde{R}_1'^{\alpha}(\beta_c) = 3g' - 2 + 3g' - 3 = 6g' - 5$$

the isotopy $\widehat{R}_1'^{\alpha}(\beta_c) \rightsquigarrow \widetilde{R}_1'^{\alpha}(\beta_c)$ can be extended such that $\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c)$ is a 1-dimensional submanifold of $\widehat{V}_{g'}$. Let $p \in \widetilde{R}_1'^{\alpha}(\beta_c) \pitchfork \widehat{R}_2'$ be a point with intersection number 1 and let

$$T_p\widehat{R}(F^{g'}) = T_p\widetilde{R}_1'^{\alpha}(\beta_c) \oplus T_p\widehat{R}_2'$$

be satisfied for the oriented tangent vector spaces. Further let U be a vector space with $\dim U = 3g' - 4$ such that

$$T_p\widehat{R}(F^{g'}) = T_p\widehat{V}_{g'} \oplus U$$

holds for the oriented vector spaces. Orient the 1-dimensional vector space $\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c) \subset \widetilde{R}_1'^{\alpha}(\beta_c)$ such that

$$T_p\widetilde{R}_1'^{\alpha}(\beta_c) = T_p\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c) \oplus U$$

is satisfied. It follows

$$T_p\widehat{R}(F^{g'}) = T_p\widetilde{R}_1'^{\alpha}(\beta_c) \oplus T_p\widehat{R}_2' = T_p\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c) \oplus U \oplus T_p\widehat{R}_2' = T_p\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c) \oplus T_p\widehat{R}_2' \oplus U = T_p\widehat{V}_{g'} \oplus U$$

as equation of oriented vector spaces. Note that the third equality holds because $(3g' - 3)(3g' - 4) \equiv 0 \mod (2)$. Considering lemma 4.1.3 this proves the statement. ■

Let p be the projection

$$p : V_{g'} \to (L_0, M_0) \; .$$

Because L_0 and M_0 commute the induced map $\widehat{p}$ projects the equivalence classes of non-abelian representations onto the pillow case:

$$\widehat{p} : \widehat{V}_{g'} \to \mathcal{PC}$$

Remark 4.4.2. 1. Let r be an element of the reducibles of $R(H_1^{g'} - k_c')$. Then by the dimensions of the tangent spaces the intersection $T_rV_{g'} \cap T_rR'^{\alpha}(\beta_c)$ is an at least 4-dimensional vector space. If there is an r with a 4-dimensional Zariski tangent space while keeping L_0 (i.e. $\mathrm{tr}\,(L_0)$) fixed, the image $\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c))$ is a curve on the pillow case which starts at the reducibles of $R(H_1^{g'} - k_c')$. Theorem 4.4.10 below tells us that $\Delta_{k,k_\pm}(e^{2i\alpha}) \neq 0$ is a sufficient condition for the existence of a projection curve. Here we denote the knot which arises from k_c due to the sign of the ± 1-surgery by $k_\pm$. (Note that previously k_+ was used for the result of *both* cases of surgeries.) To proceed with the argumentation let us assume the existence of the projection curve for the moment.

2. The example of the $(2,n)$-torus knots shows that in the case $\alpha = \pi/2$ the projection curve may run into the cone points of the pillow case (see Fig.4.8). We avoid such problems by restricting the angles α of the matrices representing the knot meridians to $I_{\pi/2} := (0,\pi) - \{\pi/2\}$. Since $\Delta_{k\subset\Sigma}(-1) \neq 0$ (see [HK98], Rem.4.9), the value of $s^\alpha(k \subset \Sigma)$ does not change in a small neighborhood of $\alpha = \pi/2$. Therefore the restriction to $I_{\pi/2}$ implies no limitation for the computation of $s^\alpha(k \subset \Sigma)$.

Definition 4.4.3. *Let $\widehat{p}^\alpha(\beta_c)$ be the projection $\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c))$ and $\alpha \in (0,\pi) - \{\pi/2\} =: I_{\pi/2}$ with $\Delta_{k,k_\pm}(e^{2i\alpha}) \neq 0$ be given. Then $\widehat{p}(\beta_c)$ is called the* projection curve *of the plat β_c on the pillow case. It consists of two branches each of which corresponds to a sheet of the 2-fold covering* SU(2) $\to$ SO(3). *The branches of the curve*

- *start at $(\frac{1}{2}\arg\lambda_0^\alpha(k,k_\pm), -2\alpha) \in \lim_{\varphi\to 0}\widehat{p}^\alpha(\beta_c)$ and $(\pi - \frac{1}{2}\arg\lambda_0^\alpha(\beta_c), 2\alpha) \in \lim_{\varphi\to 0}\widehat{p}^\alpha(\beta_c)$ respectively where (compare Th.4.4.10)*

$$\lambda_0^\alpha(k,k_\pm) = \frac{\Delta_{k_\pm\subset\Sigma}(t) - t^{\pm}\Delta_{k\subset\Sigma}(t)}{\Delta_{k_\pm\subset\Sigma}(t) - t^{\mp}\Delta_{k\subset\Sigma}(t)} \in S^1 \quad \text{and} \quad t = e^{2i\alpha} \ ,$$

- *run symmetrically with respect to $(\lim_{\varphi\to\pi}\widehat{p}^\alpha(\beta_c), 0) = (\pi/2, 0) \in \mathcal{PC}$ (provided that the same isotopies are chosen for both branches).*

Remark 4.4.4. 1. Because the intersection $\widehat{R}_1'^\alpha(\beta_c) \cap \widehat{R}_2$ is transversal at the reducibles no isotopy is required in a sufficiently small neighborhood of the endpoints of $\widehat{p}^\alpha(\beta_c)$.

2. Because of the dependence from the isotopy $\widehat{R}_1'^\alpha(\beta_c) \rightsquigarrow \widetilde{R}_1'^\alpha(\beta_c)$, the projection curve itself is not a knot invariant. It turns out that not even the endpoints of $\widehat{p}^\alpha(\beta_c)$ (whose neighborhoods are not affected by the isotopies) are knot invariants. Actually they also depend on the crossing which is changed (i.e. from the changed knot $k_\pm$, compare Th.4.4.10). Of course, theorem 4.4.10 shows that the endpoints are independent of the *plat presentation* of k'.

From remark 4.2.2 follows immediately that $\widehat{p}(\widehat{R}_2)$ is also a curve on the pillow case. Therefore we have to compare the algebraic intersection numbers of the representation spaces with that of the projection curves on the pillow case.

Lemma 4.4.5. *Let $\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c))$ be a curve on the pillow case. Then*

$$\langle \widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c), \widehat{R}_2' \rangle_{\widehat{V}_{g'}} = \langle \widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c)), \widehat{p}(\widehat{R}_2') \rangle_{\mathcal{PC}} \ .$$

Proof. Let $u \in (\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c)) \pitchfork \widehat{R}_2'$ with intersection number 1 be given and denote its image on the pillow case by $v = \widehat{p}(u)$. Regarding the normal bundle $\widehat{p}^*$ there exists an oriented vector space W with dimension $\dim W = 3g' - 4$ and

$$T_u\widehat{V}_{g'} = \widehat{p}^*[T_v\mathcal{PC}] \oplus W \ .$$

As an oriented vector space, W also completes the normal bundle of $\widehat{p}(\widehat{R}_2')$:

$$T_v\widehat{R}_2' = \widehat{p}^*[T_v\widehat{p}(\widehat{R}_2')] \oplus W \ .$$

It follows

$$\begin{aligned} T_u(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c)) \oplus T_u\widehat{R}_2' &= \widehat{p}^*[T_v\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c))] \oplus \widehat{p}^*[T_v\widehat{p}(\widehat{R}_2')] \oplus W \\ &= \widehat{p}^*[T_v\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^\alpha(\beta_c)) \oplus T_v\widehat{p}(\widehat{R}_2')] \oplus W = \widehat{p}^*[T_v\mathcal{PC}] \oplus W \ , \end{aligned}$$

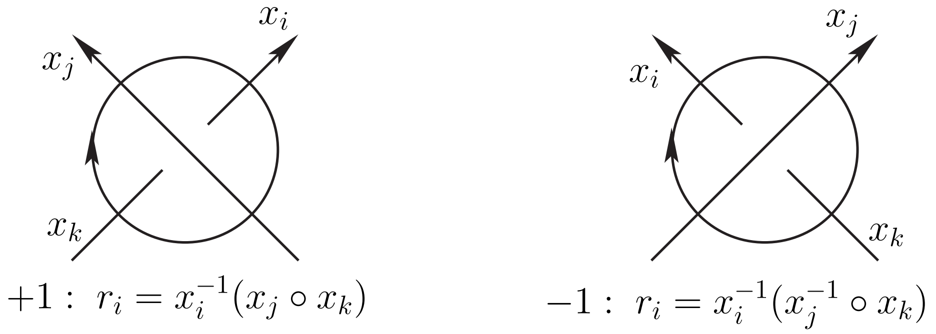


Figure 4.14: The Wirtinger relations corresponding to the two kinds of crossings.

where the second equality holds because $\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c))$ is assumed to be a curve on the pillow case. Finally we obtain

$$T_v\mathcal{PC} = T_v\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c)) \oplus T_v\widehat{p}(\widehat{R}_2')$$

as an equation of oriented vector spaces which proves the lemma. ■

To calculate $s^\alpha(k \subset \Sigma)$ with the help of the projection curve $\widehat{p}^\alpha(\beta_c)$ we use the techniques which were successfully applied in the case of the left handed $(2,n)$-torus knots. For those we used a $+1$-surgery to change a negative crossing (according to the sign of the Wirtinger relation, see Fig.4.14) into a positive.

The treatment of an arbitrary knot $k \subset \Sigma$ starts with $k_c' \subset H_2^{g'}$ in computational position. Then we switch a negative (positive) crossing which is isolated in the additional handle by a $+1$- (-1-) surgery along this handle. We arrive at the equivalent embedded situation by adding a 2π-left (2π-right) twist to k (compare Fig.4.6). Therefore we are able to change all crossings of k by means of ± 1-surgery along the additional handle.

From lemma 4.2.1 we deduce that the projection of $\widehat{i}^{\pm\frac{1}{n}}(\widehat{R}_2')$ is a curve with slope $\mp\frac{1}{n}$ (where $\pm\frac{1}{n}$ now refers to the $\pm\frac{1}{n}$-surgery along the meridian of the *additional* handle). In analogy with remark 4.2.2 we denote these curves by

$$\widehat{h}_2^{\pm\frac{1}{n}} := \widehat{p}(\widehat{i}_2^{\pm\frac{1}{n}}(\widehat{R}_2')) \subset \mathcal{PC} \quad \text{with} \quad \widehat{h}_2^{\pm} \quad \text{shorthand for} \quad \widehat{h}_2^{\pm 1} \; .$$

Then we obtain from the lemmata 4.4.1 and 4.4.5:

Theorem 4.4.6. *Let $k_c' \subset H_1^{g'}$ be a knot in computational position and let $\alpha \in I_{\pi/2}$ with $\Delta_{k,k_\pm \subset \Sigma}(e^{2i\alpha}) \neq 0$ be given. Then there exists a projection curve $\widehat{p}^\alpha(\beta_c)$ with*

$$s^\alpha(k \subset \Sigma) = (-1)^{g'} \langle \widehat{p}^\alpha(\beta_c), \widehat{h}_2 \rangle_{\mathcal{PC}} \; .$$

Because the dimension of the manifold $\widehat{i}_2(\widehat{R}_2') \cap \widehat{i}_2^{\pm}(\widehat{R}_2')$ is $3g'-6$ and provided that $\Delta_{k,k_\pm \subset \Sigma}(e^{2i\alpha}) \neq 0$, we can choose an isotopy $\widehat{R}_1'^{\alpha}(\beta_c) \rightsquigarrow \widetilde{R}_1'^{\alpha}(\beta_c)$ where $\widetilde{R}_1'^{\alpha}(\beta_c)$ is simultaneously transversal to $\widehat{i}_2(\widehat{R}_2')$ *and* $\widehat{i}_2^{\pm}(\widehat{R}_2')$. The discussed example of $(2,n)$-torus knots shows that the condition $\Delta_{k,k_\pm \subset \Sigma}(e^{2i\alpha}) \neq 0$ is not necessary.

Corollary 4.4.7. *Suppose that the assumptions of theorem 4.4.6 hold. Then there exists a projection curve $\widehat{p}^\alpha(\beta_c)$ with*

$$\Delta s^\alpha(k \subset \Sigma) = s^\alpha(k_\pm \subset \Sigma) - s^\alpha(k \subset \Sigma) = \pm(-1)^{g'} \langle \widehat{p}^\alpha(\beta_c), \widehat{h}_2^{\pm} - \widehat{h}_2 \rangle_{\mathcal{PC}}$$

Proof. Choose an isotopy $\widehat{R}_1'^{\alpha}(\beta_c) \rightsquigarrow \widetilde{R}_1'^{\alpha}(\beta_c)$ as above. Using theorem 4.4.6 and regarding the orientations of the branches of the projection curve we obtain

$$s^{\alpha}(k \subset \Sigma) - s^{\alpha}(k_- \subset \Sigma) = (-1)^{g'} \langle \widehat{p}^{\alpha}(\beta_c), \widehat{h}_2^- - \widehat{h}_2 \rangle_{\mathcal{PC}} ,$$

and

$$s^{\alpha}(k_+ \subset \Sigma) - s^{\alpha}(k \subset \Sigma) = (-1)^{g'} \langle \widehat{p}^{\alpha}(\beta_c), \widehat{h}_2^+ - \widehat{h}_2 \rangle_{\mathcal{PC}} .$$

■

The open oriented arcs $\widehat{h}_2^{\pm}$ and $-\widehat{h}_2$ together with the edge $(0,\pi) \times \{\pi\}$ of the pillow case build the oriented difference cycles

$$C_{\pm} := \widehat{h}_2^{\pm} \cup -\widehat{h}_2 \cup (0,\pi) \times \{\pi\} \subset \mathcal{PC} .$$

Therefore we can write

$$\Delta s^{\alpha}(k \subset \Sigma) = \pm(-1)^{g'} \langle \widehat{p}^{\alpha}(\beta_c), C_{\pm} \rangle_{\mathcal{PC}} .$$

Remark 4.4.8. Let $\tilde{g}$ denote the genus of the Heegaard splitting after stabilizing the original splitting with genus g. Then we have to switch the orientation of difference cycle (or equivalently of the projection curve of the knot) if and only if $g - \tilde{g} \equiv 1 \mod (2)$ (compare the proof of Lem.4.1.3).

For the change of the intersection number $s^{\alpha}(k \subset \Sigma)$ we now obtain:

Theorem 4.4.9. *Let the projection curve of corollary 4.4.7 and the difference cycles $C_{\pm}$ as above be given. Then*

$$\Delta s^{\alpha}(k \subset \Sigma) = s^{\alpha}(k_{\pm} \subset \Sigma) - s^{\alpha}(k \subset \Sigma) \quad = \begin{cases} \mp 1, & \text{iff } \widehat{p}^{\alpha}(\beta_c) \text{ ends in } C_{\pm} \\ 0, & \text{iff } \widehat{p}^{\alpha}(\beta_c) \text{ does not end in } C_{\pm} \end{cases} .$$

Proof. Remark 4.4.8 force us to change the orientation of $\widehat{p}^{\alpha}(k_c')$ (or alternatively that of $C_{\pm}$) according to $(-1)^{g'}$ where g' denotes the genus of the computational Heegaard splitting chosen. Therefore the differences $s^{\alpha}(k_{\pm} \subset \Sigma) - s^{\alpha}(k \subset \Sigma)$ are independent of g'. If we fix the orientations by choosing $s^{\pi/2}(k_3 \subset S^3) = -1$ for the *lefthanded* trefoil (see equation (4.10)) the statement follows from corollary 4.4.7. ■

The endpoints of the projection curve are calculated in the next theorem. At the same time this will ensure the existence of the projection curve as it was anticipated in definition 4.4.3.

As mentioned in remark 4.4.4 there is no need to isotope in a sufficiently small neighborhood of the reducibles of $\pi_1(H_1^{g'} - k_c')$. Therefore the endpoints of the projection curve and their relation to $\widehat{h}_2$ provide direct (i.e. independent from the isotopies chosen on compact subsets) information on the representation space of $\Sigma - k$. Anyway the information is sufficient to determine $s^{\alpha}(k \subset \Sigma)$.

Theorem 4.4.10. *Let $k_c' \subset H_1^{g'}$ be a knot in computational position and let $\alpha \in I_{\pi/2}$ with $\Delta_{k,k_{\pm}}(e^{2i\alpha}) \neq 0$ be given. Then the projection $\widehat{p}^{\alpha}(k_c') := \widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c))$ is a curve on the pillow case. The projection curve has endpoints*

$$\lim_{\varphi \to 0} \widehat{p}^{\alpha}(\beta_c) = (\frac{1}{2} \arg \lambda_0^{\alpha}(k,k_{\pm}), -2\alpha) \cup (\pi - \frac{1}{2} \arg \lambda_0^{\alpha}(k,k_{\pm}), 2\alpha) \subset \mathcal{PC} ,$$

where

$$\lambda_0^{\alpha}(k,k_{\pm}) = \frac{\Delta_{k_{\pm} \subset \Sigma}(t) - t^{\pm} \Delta_{k \subset \Sigma}(t)}{\Delta_{k_{\pm} \subset \Sigma}(t) - t^{\mp} \Delta_{k \subset \Sigma}(t)} \in S^1 \quad \text{and} \quad t = e^{2i\alpha} .$$

Proof. To prove the existence of the projection curve we will calculate the reducible limits of $\widehat{p}(\widehat{V}_{g'} \pitchfork \widetilde{R}_1'^{\alpha}(\beta_c))$ (cf. Rem.4.4.2). First we discuss the case of +1-surgery from which the results for the −1-surgery will follow immediately. Without loss of generality we consider the situation of two strands which perform a 3π-left twist while running through the additional handle after the +1-surgery was applied. This is already given if the crossing we intend to alter is positive in the sense above (compare Fig.4.15). Otherwise we establish the situation by a Reidemeister move of type II.

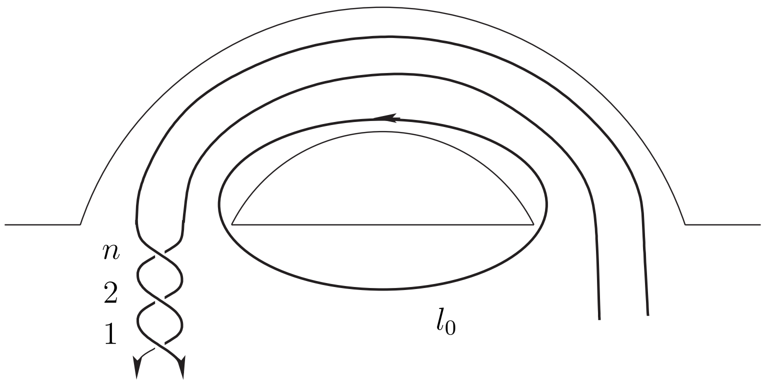


Figure 4.15: The enumeration of the crossings of k'_c in the additional handle.

For the computation we use the regular projection as it is given by the plat presentation of k'_+. The projection should have n crossings. Then we obtain the fundamental group $\pi_1(H_1^{g'} - k'_+)$ from the generators and relations as they are given by a Wirtinger presentation (see [BZ85], Ch.3B). We enumerate the three considered crossings of k'_+ as shown in figure 4.15. From the three crossings we obtain the Wirtinger relations

$$\begin{array}{rrcl} 1: & l_0^{-1} \circ x_1 & = & (l_o^{-1} \circ x_2) \circ x_n = l_0^{-1} x_2 l_0 x_n l_0^{-1} x_2^{-1} l_0 \Leftrightarrow x_1 = x_2 l_0 x_n l_0^{-1} x_2^{-1} \\ 2: & l_0^{-1} \circ x_2 & = & x_n \circ x_{n-1} \Leftrightarrow x_2 = l_0 x_n x_{n-1} x_n^{-1} l_0^{-1} \\ n: & x_n & = & x_{n-1} \circ x_{n-2} \,. \end{array}$$

By a little abuse of notation let $(\mathrm{d}\lambda_i^{-1}, \mathrm{d}L_i^{-1}) \in su(2) = \mathbb{R} \oplus \mathbb{C}$, $0 \leq i \leq g$, and $(0, \mathrm{d}X_i) \in su(2)$, $1 \leq i \leq n$, denote the tangent vectors of the Zariski tangent space at the reducible representation corresponding to the $g+1$ longitudes of $H_1^{g'}$ and to the generators of the n meridians respectively. Note that $\mathrm{d}M_0 = \mathrm{d}(X_1 X_2)$ holds providing that a suitable orientation for k' is chosen. From the proof of theorem 3.4.2 follows $\mathrm{d}L_i = \mathrm{d}1 = 0$ and $\mathrm{d}\lambda_i = 0$, $1 \leq i \leq g$, for the tangent vectors of the remaining longitudes.
Derived from the conjugated knot meridians by means of Fox calculus the corresponding relations determining the Zariski tangent space at the reducibles of $R^{\alpha}(H_1^{g'} - k'_+)$ keep the decomposition of $su(2)$ into a real and a complex subspace. Thus we obtain $0 \cdot \mathrm{d}\lambda_0^{-1} = 0 \in \mathbb{R}$ for the real and the

following matrix equation for the complex part:

$$\begin{pmatrix} t^2-t & -\lambda_0^{-1} & \lambda_0^{-1}(1-t) & 0 & \dots & 0 & 0 & 0 & t \\ t-1 & 0 & -\lambda_0^{-1} & 0 & \dots & 0 & 0 & t & 1-t \\ 0 & 0 & 0 & 0 & \dots & 0 & t & 1-t & -1 \\ 0 & * & * & * & \dots & * & * & * & * \\ \vdots & & & \dots & & & & \vdots & \vdots \\ 0 & * & * & * & \dots & * & * & * & * \end{pmatrix} \begin{pmatrix} \mathrm{d}L_0^{-1} \\ \mathrm{d}X_1 \\ \mathrm{d}X_2 \\ \mathrm{d}X_3 \\ \vdots \\ \mathrm{d}X_n \end{pmatrix} = 0 \in \mathbb{C} \tag{4.14}$$

which we abbreviate by $M_{k_+}\begin{pmatrix}\mathrm{d}L_0^{-1}\\ \mathrm{d}\boldsymbol{X}\end{pmatrix} = 0$, $M_{k_+} \in \mathbb{C}^{n \times n+1}$. The notation is justified since the tangent spaces of $R(\Sigma - k_+)$ and $R(S^3 - k'_+)$ are equal at the reducibles. Each conjugating meridian $x_i^{\pm 1}$ contributes a factor $t^{\pm 1} = e^{\pm 2i\alpha} \in S^1$ which provides a rotation in the tangent plane isomorphic to $\mathbb{C}$. Further, each appearance of the character L_0^{-1} in the derived words contributes a factor denoted by $\lambda_0^{\alpha}(k, k_+)^{-1} \in S^1$ (cf. Rem.2.5.4) where the dependency from k, k_+ and α is omitted. By setting $\lambda_0^{-1} = 1$ and deleting the first column of $M(k_+) = (c_0, c_1, \dots, c_n)$, $c_i \in \mathbb{C}^n$, we obtain an Alexander matrix $A_{k'_+}$ of the knot $k'_+ \subset S^3$, i.e.

$$A_{k'_+} = (c_1(\lambda_0^{-1} = 1), c_2(\lambda_0^{-1} = 1), c_3, \dots, c_n)\ .$$

By the computational position of the knot we have

$$c_1^{(1)} = \begin{pmatrix} t-1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} = (t-1)\begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} =: (t-1)e_1 \in \mathbb{C}^{n-1}\ ,$$

where $c_i^{(j)}$ denotes the column vector c_i with the j-th entry deleted. Writing M_{k_+} as a n-dimensional vector of its rows $r_i \in \mathbb{C}^{n+1}$, $M_{k_+} = \begin{pmatrix} r_1 \\ \vdots \\ r_n \end{pmatrix}$, we add $-t$ times r_2 to r_1:

$$\mathbb{C}^{n\times n+1} \ni M'_{k_+} = \begin{pmatrix} 0 & -\lambda_0^{-1} & \lambda_0^{-1} & 0 & \dots & 0 & 0 & -t^2 & t^2 \\ t-1 & 0 & -\lambda_0^{-1} & 0 & \dots & 0 & 0 & t & 1-t \\ 0 & 0 & 0 & 0 & \dots & 0 & t & 1-t & -1 \\ 0 & * & * & * & \dots & * & * & * & * \\ \vdots & & & & \dots & & & \vdots & \vdots \\ 0 & * & * & * & \dots & * & * & * & * \end{pmatrix}.$$

This simplifies the calculations and of course preserves the solution of (4.14).
Considering the equations for M'_{k_+} and because the angle of the conjugating matrix L_0 is arbitrary at the reducibles we obtain

$$\mathrm{d}\lambda_0^{-1} \in \mathbb{R}\ ,\ \mathrm{d}X_i = \mathrm{d}X \in \mathbb{C}, 1 \le i \le n\ ,\ \mathrm{d}L_0^{-1} = (\lambda_0^{-1} - 1)/(t-1)\mathrm{d}X \in \mathbb{C}\ .$$

Thus, taking the SO(3)-quotient and projecting onto the pillow case, the projecting curve of the reducibles can be identified with $[0, \pi] \times \{\pm 2\alpha\} \subset \mathcal{PC}$.
Because the real part provides no further information we have to solve (4.14) to proceed in the determination of λ_0. If the solution for λ_0 is connected uniquely (up to multiplication in $\mathbb{C}$) with

a $\mathrm{d}\mathbf{X} \in \mathbb{C}^n$, the 2-dimensional complex vector space (and therefore the 4-dimensional real vector called for in Rem.4.4.2) is spanned by $\mathrm{d}\mathbf{X}$ and $(1,\ldots,1) \in \mathbb{C}^n$.
By SU(2)-conjugation we again choose $X_1 = (\alpha, c_\varphi \boldsymbol{e}_x + s_\varphi \boldsymbol{e}_y)$ and $X_2 = (\alpha, \boldsymbol{e}_x)$ for the matrices representing the knot meridians x_1 and x_2 of the additional handle. This induces $\mathrm{d}X_1 \in \mathbb{R}$ and $\mathrm{d}X_2 = 0$ and we thus have to solve the following systems of equations

$$M_{k_+} \begin{pmatrix} \mathrm{d}L_0^{-1} \\ \mathrm{d}X_1 \\ 0 \\ \mathrm{d}\boldsymbol{X} \end{pmatrix} = M'_{k_+} \begin{pmatrix} \mathrm{d}L_0^{-1} \\ \mathrm{d}X_1 \\ 0 \\ \mathrm{d}\boldsymbol{X} \end{pmatrix} = 0 \ , \ \ \mathrm{d}L_0^{-1} \in \mathbb{C} \ , \mathrm{d}X_1 \in \mathbb{R} \ , \ \mathrm{d}\boldsymbol{X} \in \mathbb{C}^{n-2} \ . \tag{4.15}$$

From M_{k_+} and M'_{k_+} follows the Alexander polynomial of $k'_+ \subset S^3$ (and therefore of $k_+ \subset \Sigma$) by taking the determinant of the matrices with the first row as well as the first and third column deleted:

$$\Delta_{k_+}(t) = \pm t^s \det(c_1^{(1)}, c_3^{(1)}, \ldots, c_n^{(1)}) =: \pm t^s \tilde{A}_{k'_+} \ .$$

The factor $\pm t^s$, $s \in \mathbb{Z}$, is necessary to obtain a symmetric polynomial. We are forced to delete the third column by the condition $\mathrm{d}X_2 = 0$; the deletion of the first row is done in order to achieve independence of λ_0. Deleting of the rows r_2 and r_3 together with the columns c_{n-1} and c_n yields

$$\Delta_k(t) = \mp t^{s-1} \det(c_1^{(1,2,3)}, c_3^{(1,2,3)}, \ldots, c_{n-2}^{(1,2,3)}) =: \mp t^{s-1} \det \tilde{A}_{k'} \ .$$

The exponent $s-1$ is due to the fact that the projection of k_+ possesses two more crossings than k. The additional sign can be derived by setting $t=1$ in the reduced matrices and keeping in mind that $\Delta_{k\subset\Sigma}(1) = 1$ holds for the Alexander polynomial. Note that, despite of the relation to the Alexander polynomials, it is not possible to obtain the Alexander matrix of k' directly as a reduced matrix from M_{k_+} or M'_{k_+}.
Setting $\mathrm{d}X_1 = 1 \in \mathbb{R}$ the first equation in (4.15) yields:[4]

$$\lambda_0^{-1} = t^2(\mathrm{d}X_n - \mathrm{d}X_{n-1}) \ .$$

To solve the equations in (4.15) which will determine $\mathrm{d}X_{n-1}$ and $\mathrm{d}X_n$ we note that $\tilde{A}_{k'_+}$ is invertible over $\mathbb{C}$ if $\Delta_{k'_+}(t) \neq 0$ holds. With

$$\begin{aligned} M_1 &:= (e_1, c_3^{(1)}, \ldots, c_n^{(1)}) \in \mathbb{C}^{n-1\times n-1} \ , \\ M_i &:= (c_1^{(1)}, c_3^{(1)}, \ldots, c_{i-1}^{(1)}, e_1, c_{i+1}^{(1)}, \ldots, c_n^{(1)}) \in \mathbb{C}^{n-1\times n-1}, \ i = 3,\ldots,n \ , \end{aligned}$$

we obtain from Cramer's rule

$$e_1 = \frac{\det M_1}{\det \tilde{A}_{k'_+}} c_1^{(1)} + \sum_{i=3}^{n} \frac{\det M_i}{\det \tilde{A}_{k'_+}} c_i^{(1)} \ . \tag{4.16}$$

From the second row of (4.15), with $\mathrm{d}X_1 = 1$, we have:

$$(1-t)\mathrm{d}L_0^{-1} = c_1^{\{2\}} + \sum_{i=1}^{n} c_i^{\{2\}} \mathrm{d}X_i \ , \tag{4.17}$$

[4] Another choice multiplies a constant factor to all tangent vectors. Therefore the angle of λ_0, we are concerned with, is not affected by this choice.

where $c_i^{\{j\}}$ denotes the j-th component of c_i. On the other hand we know from the first row of (4.16)

$$1 = \frac{\det M_1}{\det \tilde{A}_{k'_+}} c_1^{\{2\}} + \sum_{i=3}^{n} \frac{\det M_i}{\det \tilde{A}_{k'_+}} c_i^{\{2\}} \ . \tag{4.18}$$

Because the solution of (4.16) is unique under the assumption $\Delta_{k_+}(t) \neq 0$ we obtain

$$\mathrm{d}L_0^{-1} = \frac{\det \tilde{A}_{k'_+}}{(1-t)\det M_1} \quad \text{and} \quad \mathrm{d}X_i = \frac{\det \tilde{A}_{k'_+}}{\det M_1} \ , \ i = 3, \ldots, n \ , \tag{4.19}$$

by comparing (4.17) and (4.18). Note that we have $1 - t \neq 0$ because of $\alpha \in (0, \pi)$. With $\lambda_0 = t^2(\mathrm{d}X_n - \mathrm{d}X_{n-1})$ equation (4.19) yields:

$$\lambda_0^{-1} = \frac{t^2}{\det M_1}(\det M_n - \det M_{n-1}) \ . \tag{4.20}$$

To compute the matrices M_{n-1} and M_n we expand them with respect to the first row. For M_{n-1} follows

$$M_{n-1} = \left(\begin{array}{cccc|cc} 0 & 0 & \cdots & 0 & 1 & 1-t \\ 0 & 0 & \cdots & t & 0 & -1 \\ \hline & & & & 0 & 0 \\ & & & & \vdots & \vdots \\ & A' & & \tilde{A}_{k'} & 0 & 0 \end{array} \right) \Rightarrow \det M_{n-1} = -\det \tilde{A}_{k'} = \pm t^{s-1}\Delta_k(t) \ , \tag{4.21}$$

which is independent from the number of crossings n. The matrix

$$A' := (c_1^{(1,2,3,4)}, c_3^{(1,2,3,4)}, \ldots, c_n^{(1,2,3,4)}) \in \mathbb{C}^{n-4 \times n-4}$$

defined above will be used while computing $\det M_n$. Laplace expanding of $\det M_n$ provides:

$$\det M_n = (t-1)\det \tilde{A}_{k'} - t \det A' \ , \tag{4.22}$$

which is again independent of n. To compute $\det A'$ we note that

$$\det \tilde{A}_{k'_+} = t \det M_{n-1} + (1-t)\det M_n = \pm t^s \Delta_{k_+}(t) \ ,$$

which by (4.22) leads to

$$\det A' = \frac{\mp t^{s-1}(-t^2 + t - 1)\Delta_k - \pm t^s \Delta_{k_+}}{t(1-t)} \ . \tag{4.23}$$

Deleting the second row of $\tilde{A}_{k'_+}$ instead of the first and defining the matrices

$$\widetilde{M}_1 := (e_1, c_3^{(2)}, \ldots, c_n^{(2)}) \quad \text{and} \quad \widetilde{M}_i := (c_1^{(2)}, c_3^{(2)}, \ldots, c_{i-1}^{(2)}, e_1, c_{i+1}^{(2)}, c_n^{(2)}) \ , i = 3, \ldots, n \ ,$$

analogously to the M_i above, we obtain

$$\mp t^s \Delta_{k_+}(t) = -\det \widetilde{M}_1 + t \det \widetilde{M}_n \ .$$

The factor $\mp t^s$ appears because $(-1)^{1+2} = -(-1)^{2+2}$ holds and crossing I is positive (as crossing II). From $\widetilde{M}_1 = M_1$ and $\widetilde{M}_n = M_n$ follows

$$\det M_1 = t \det M_n \pm t^s \Delta_{k_+} \tag{4.24}$$

and the equations (4.22) and (4.23) imply:

$$\det M_n = \frac{\pm t^s(\Delta_{k_+} - \Delta_k)}{1-t} \ . \tag{4.25}$$

Further we obtain from (4.21) and (4.25):

$$\det M_n - \det M_{n-1} = \frac{\pm t^{s-1}(t\Delta_{k_+} - \Delta_k)}{1-t} \ . \tag{4.26}$$

Equation (4.25) together with (4.24) yields

$$\det M_1 = \frac{\pm t^{s+1}(t^{-1}\Delta_{k_+} - \Delta_k)}{1-t}$$

where the assumptions imply $\det M_1 \neq 0$. Finally the use of equation (4.20) leads to

$$\lambda_0^{-1} = \frac{\Delta_k - t\Delta_{k_+}}{\Delta_k - t^{-1}\Delta_{k_+}} \ . \tag{4.27}$$

Because the corresponding Alexander matrix for $\lambda_0 = 1$ is that of k'_+ it is more convenient to interpret k_+ as k and k as k_- respectively. So formula (4.27) becomes

$$\lambda_0^{-1}(k, k_-) = \frac{\Delta_{k_-}(t) - t\Delta_k(t)}{\Delta_{k_-}(t) - t^{-1}\Delta_k(t)} \ . \tag{4.28}$$

Because $\lambda_0^{-1}(k, k_\pm)$ is related to the rotation induced by conjugation with $\pm L_0^{-1}$, for the λ-values of the endpoints of the projection curve $\widehat{p}^\alpha(\beta_c)$ immediately follows

$$\lim_{\varphi \to 0} \lambda(\widehat{p}^\alpha(\beta_c)) = \frac{1}{2} \arg \lambda_0^\alpha(k, k_\pm) \cup (\pi - \frac{1}{2} \arg \lambda_0^\alpha(k, k_\pm)) \ .$$

Here the first (the second) term is related to the conjugation with L_0^{-1} (with $-L_0^{-1}$) and therefore yields the value of the endpoint related to $\{-2\alpha\}$ (to $\{2\alpha\} \subset \mathcal{PC}$ resp.). To determine the corresponding endpoints on the pillow case we have to consider the chosen parameterization. In the case $\alpha > \pi/2$ this may interchange $\pm 2\alpha$ due to the fundamental action on the pillow case (see Rem.2.3.5 and Lem.4.4.13). Thus the first part of the statement is proved.

If we add a 2π-right twist to k'_c we can establish the same situation as shown in figure 4.15 with the positive crossings substituted by negative ones. Then the calculation is similar to that for adding a 2π-left twist to the mirror image k'^*_c of k'_c. This will cause a substitution of k_- by k^*_+ and t by t^{-1}. Because $\Delta_{k^*}(t) = \Delta_k(t^{-1}) = \Delta_k(t)$ holds for the Alexander polynomials we obtain the stated formula for $\lambda_0^\alpha(k, k_+)$. ■

Example 4.4.11. For the left-handed trefoil $k = k_{3_1}$ we find $k_- = k_0$ and $k_+ = k_{5_1}$. The formulas of theorem 4.4.10 yield

$$\lambda_0^\alpha(k_{3_1}, k_0) = -t^{-3} = \lambda_0^\alpha(k_{3_1}, k_{5_1}) \Rightarrow \lim_{\varphi \to 0} \widehat{p}^\alpha(\sigma_1^3) = \frac{1}{2} \arg \lambda_0^\alpha(k_{3_1}) \cup \pi - \frac{1}{2} \arg \lambda_0^\alpha(k_{3_1}) = \frac{\pi}{2} - 3\alpha \cup \frac{\pi}{2} + 3\alpha \ .$$

This result is consistent with theorem 4.2.5 which shows the representation curves having constant slope $\frac{1}{n}$ and therefore endpoints being on a strait line with constant slope.

It should be emphasized that we obtain equal results from computing the endpoints by $\lambda_0(k,k_+)$ and $\lambda_0(k,k_-)$. This is the case because both determine the endpoints of the *same* projection curve on the pillow case namely $\widehat{p}^\alpha(\beta_c)$.

Remark 4.4.12. Because $\Delta_{k,k_\pm}(1)=1$, for the limits $\alpha\to 0,\pi$ we obtain:

$$\lim_{\alpha\to 0,\pi}\lambda_0(k,k_\pm)=-1\ ,$$

which leads to the common end point of both branches:

$$\lim_{\alpha\to 0,\pi}\widehat{p}^\alpha(\beta_c)=\lim_{\alpha\to 0,\pi}(\arg(-1)/2,2\alpha)=\lim_{\alpha\to 0,\pi}(\arg(-1)/2,-2\alpha)=(\frac{\pi}{2},0)\in\mathcal{PC}\ .$$

For the computation of $s^\alpha(k\subset\Sigma)$ we will make use of the following observation.

Lemma 4.4.13. *Let $t=e^{2i\alpha}$, $\alpha\in I_{\pi/2}$, be given. Consider the maps $f_\pm:\mathbb{R}\cup\{\infty\}\to S^1\subset\mathbb{C}$, $f_\pm(a)=\frac{a-t^{\pm1}}{a-t^{\mp1}}$. Then:*

$$\alpha<\pi/2:\ \arg f_+(a)\in\begin{cases}(0,4\alpha) & \text{iff } a<0\\ (4\alpha,2\pi) & \text{iff } a>0\end{cases},\ \arg f_-(a)\in\begin{cases}(2\pi-4\alpha,2\pi) & \text{iff } a<0\\ (0,2\pi-4\alpha) & \text{iff } a>0\end{cases},$$

$$\alpha>\pi/2:\ \arg f_+(a)\in\begin{cases}(4\alpha-2\pi,2\pi) & \text{iff } a<0\\ (0,4\alpha-2\pi) & \text{iff } a>0\end{cases},\ \arg f_-(a)\in\begin{cases}(0,4\pi-4\alpha) & \text{iff } a<0\\ (4\pi-4\alpha,2\pi) & \text{iff } a>0\end{cases}.$$

Proof. The Moebius transformation f_+ maps the real line onto the circle $S^1\subset\mathbb{C}$. Because $f_+(0)=t^2$, $f_+(1)=-t$ and $f_+(\infty)=1$ the statement follows for f_+. Analogous arguments will prove it for f_-. ∎

Corollary 4.4.14. *Let $k,k_\pm\subset\Sigma$ with $\Delta_{k,k_\pm\subset\Sigma}(t)\neq 0$, $t=e^{2i\alpha}$, $\alpha\in I_{\pi/2}$, and a projection curve $\widehat{p}^\alpha(\beta_c)$ which is simultaneously transversal to $\widehat{h}_2$ and $\widehat{h}_2^\pm$ be given. Then*

$$s^\alpha(k_\pm\subset\Sigma)-s^\alpha(k\subset\Sigma)=\begin{cases}\mp1 & \text{iff}\quad \Delta_{k_\pm}(t)/\Delta_k(t)<0\\ 0 & \text{iff}\quad \Delta_{k_\pm}(t)/\Delta_k(t)>0\end{cases}$$

Proof. Setting $a:=\Delta_{k_\pm}(t)/\Delta_k(t)$ theorem 4.4.10 and lemma 4.4.13 yield

$$\widehat{p}^\alpha(\beta_c)\begin{cases}\text{ends in } C_\pm \ \text{iff } \Delta_{k_\pm}(t)/\Delta_k(t)<0\\ \text{does not end in } C_\pm \ \text{iff } \Delta_{k_\pm}(t)/\Delta_k(t)>0\end{cases}.$$

Then, considering the fundamental action on the pillow case in the case $\alpha>\pi/2$, the statement follows from theorem 4.4.9. ∎

From the computational algorithm follows immediately

Corollary 4.4.15. *Let k be a knot in a homology 3-sphere Σ with trivial Alexander polynomial. Then $\lim_{\alpha\to 0,\pi}s^\alpha(k\subset\Sigma)=2\lambda(\Sigma)$ holds.*

Proof. For α in a sufficiently small neighborhood of 0 and π respectively, $\Delta_{k\subset\Sigma}(e^{2i\alpha})>0$ holds for *all* knots k in Σ. Considering a (finite) unknotting process for k the statement follows from corollary 4.4.14 and $s^\alpha(k_0\subset\Sigma)=2\lambda(\Sigma)$ (see Lem.3.4.8). ∎

Remark 4.4.16. Because we know that any knot is unknotable (in the sense of Th.3.2.6) sooner or later a non-trivial Alexander polynomial has to alter when switching several crossings. (If the Alexander polynomial is constant we have $s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma)$ for all α which is a trivial case for the computation.) Let us however assume that $\Delta_k = \Delta_{k_\pm}$ holds, then we have $\lambda_0(k, k_+) = \frac{1-t}{1-t^{-1}}$ and $\lambda_0(k, k_-) = \frac{1-t^{-1}}{1-t}$ respectively (see Th.4.4.10). Now from lemma 4.4.13 follows $\Delta s^\alpha(k \subset \Sigma) = 0$ for $\alpha \in (0, \pi)$ and $s^\alpha(k \subset \Sigma)$ can still be computed.

The recursive formula for the computation of $s^\alpha(k \subset \Sigma)$ stated in corollary 4.4.14 is quite similar to the recursive formula for the equivariant signature $\sigma_\omega(k \subset \Sigma)$.

Lemma 4.4.17. *Let k and $k_\pm$ be knots in Σ and $\omega \in S^1$ with $\Delta_{k,k_\pm}(\omega) \neq 0$ be given. Then*

$$\sigma_{k_\pm}(\omega) - \sigma_k(\omega) = \begin{cases} \mp 1 & \text{iff} \quad \Delta_{k_\pm}(\omega)/\Delta_k(\omega) < 0 \\ 0 & \text{iff} \quad \Delta_{k_\pm}(\omega)/\Delta_k(\omega) > 0 \end{cases} .$$

Proof. First consider a knot k in S^3. Then for the ω-signature, always an even number,

$$\sigma_k(\omega) \equiv 0 \mod 4 \quad \text{iff } \Delta_k(\omega) > 0 \quad \text{and} \quad \sigma_k(\omega) \equiv 2 \mod 4 \quad \text{iff } \Delta_k(\omega) < 0 \tag{4.29}$$

holds (see [HK98], p.497). Considering the Seifert matrices of k_+, k_- and k we obtain:

$$-2 \leq \sigma_{k_+}(\omega) - \sigma_k(\omega) \leq 0 \quad \text{and} \quad 0 \leq \sigma_{k_-}(\omega) - \sigma_k(\omega) \leq 2 . \tag{4.30}$$

Applying the arguments of theorem 2.2.8, the relations (4.29) and (4.30) hold for knots k and $k_\pm$ in Σ as well. This completes the proof. ■

Now we are ready to compute the intersection number $s^\alpha(k \subset \Sigma)$.

Theorem 4.4.18. *Let $k \subset \Sigma$ be a knot and $\alpha \in (0, \pi)$ with $\Delta_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ be given. Then*

$$s^\alpha(k \subset \Sigma) = 2\lambda(\Sigma) + \frac{1}{2}\sigma_{k\subset\Sigma}(e^{2i\alpha}) .$$

Proof. We first choose a rational $\alpha \in I_{\pi/2}$ to achieve that $e^{2i\alpha} = t$ is a transcendental number ([Lor96], Ch.17, Th.3). Consider a sequence of crossing changes, which leads from k to a knot k_0 with Alexander polynomial $\Delta_{k_0}(t) = 1$, the conditions $\Delta_{k,k_\pm}(t) \neq 0$ of corollary 4.4.14 and lemma 4.4.17 are satisfied during all steps. Because $s^\alpha(k \subset \Sigma)$ and $\sigma_{k\subset\Sigma}(e^{2i\alpha})$ are locally constant and the equation $s^\alpha(k_0 \subset \Sigma) = 2\lambda(\Sigma)$ holds for all $\alpha \in (0, \pi)$ (and the rationals are dense in $\mathbb{R}$) the statement follows by comparing corollary 4.4.14 and lemma 4.4.17. Finally, since $\Delta_{k\subset\Sigma}(-1) \neq 0$ holds for all $k \subset \Sigma$, the result is valid for *all* $\alpha \in (0, \pi)$ (see Rem.4.4.2). ■

The result of the computation will establish $s^\alpha(k \subset \Sigma)$ as a knot invariant.

Corollary 4.4.19. *Suppose that the assumptions of theorem 4.4.18 hold. Then $s^\alpha(k \subset \Sigma)$ is an invariant for knots in homology 3-spheres.*

Proof. Because $\sigma_{k\subset\Sigma}(e^{2i\alpha})$ is a knot invariant $s^\alpha(k \subset \Sigma)$ is independent of the chosen plat presentation of $k' \subset H_1^g \subset S^3$. All that's left is to ensure that our computation of $s^\alpha(k \subset \Sigma)$ is independent of the chosen Heegaard splitting. For the Casson invariant this is explicitly shown in its existence proof (see [Sav99], Ch.16.3). The independence of the contribution obtained from the knot complement follows from theorem 4.4.9. Hence we are actually free to choose the standard and therefore as well the computational position for k' which completes the proof. ■

Remark 4.4.20. For the 3-sphere S^3 we have $\lambda(S^3) = 0$ and obtain theorem 1.2 of [HK98] where $s^\alpha(k \subset S^3) = h^\alpha(k)$ (and of course Lin's result for the trace free case $\alpha = \pi/2$).

4.5 Applications

Theorem 4.4.18 together with lemma 3.4.9 immediately implies a criterion for an abelian representation ρ_α to be a limit of an non-abelian representations of $\pi_1(\Sigma - k)$.

Theorem 4.5.1. *Let $k \subset \Sigma$ be a knot in a homology 3-sphere and $\alpha \in [0, \pi]$ such that $\Delta_{k\subset\Sigma}(e^{2i\alpha}) = 0$. Then the abelian representation ρ_α is a limit of non-abelian representations of $\pi_1(\Sigma - k)$ if $\sigma_{k\subset\Sigma}$ jumps at $e^{2i\alpha}$.*

Proof. If this is not the case, theorem 4.4.18 provides a contradiction to lemma 3.4.9. ■

Combining theorem 4.4.18 with the statement above yields the following

Corollary 4.5.2. *Let $k \subset \Sigma$ be a knot in a homology 3-sphere. If there exists an $\alpha \in [0, \pi]$ such that $\sigma_{k\subset\Sigma}(e^{2i\alpha}) \neq 0$ then $\pi_1(\Sigma - k)$ admits a non-abelian* SU(2)*-representation.*

Remark 4.5.3.

1. Note that the assumptions of theorem 4.5.1 and corollary 4.5.2 are satisfied if $e^{2i\alpha}$ is a root of $\Delta_{k\subset\Sigma}(e^{2i\alpha})$ with odd degree.

2. Theorem 4.5.1 was independently proved by C. Herald using gauge theory (cf. [Her97]). His invariant $h_\alpha(\Sigma, k)$ contains the Casson invariant with a factor 4. The additional factor 2 (compared with $s^\alpha(k \subset \Sigma)$) comes from determining the intersection number with the help of the 2-fold cover of the pillow case (given by the involution of the torus). This yields (cf. [Sav02], Th.5.17)

$$h_0(\Sigma, k) = \langle r(\mathcal{R}_h(\Sigma - k)), S_0 \rangle_{\mathcal{PC}} = 4\lambda(\Sigma) = 2 \lim_{\alpha \to 0} s^\alpha(k \subset \Sigma) .$$

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