Analysis of New Physics in B Decays

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Abstract

The inclusive semileptonic decay $B \to X_c l\nu$ yields the best extracted value of $|V_{cb}|$ with an uncertainty of about 2 %, which is due to its precise theoretical description by the heavy-quark expansion (HQE) and the large amount of data collected by the B factories Babar and Belle. Besides the precision determination of $|V_{cb}|$ and the HQE parameter it also enables us to test for new physics effects. Normally decays with a small standard model contribution are investigated on the quest for new physics, like flavour-changing neutral currents, which are suppressed in the standard model by the GIM mechanism. But as right-handed weak currents are absent in the standard model their appearance would be a smoking gun signal for new physics in the decay considered here.

We will perform such an analysis, which is known form the leptonic sector as "Michel parameter analysis". Contributions from possible new physics effects will be derived with effective field theory methods, yielding an enhanced $b \to c$ current, with not only a right-handed vector contribution, but also scalar and tensor couplings. We will repeat the computations of the decay with the enhanced current up to $\mathcal{O}(1/m_{\mathrm{b}}^2)$ in the HQE and up to $\mathcal{O}(\alpha_s)$ in the perturbative corrections.

A moment analysis of the moments of the lepton energy spectrum and the hadronic invariant mass has become a reliable tool in both the theoretical evaluation and the experimental determination. This combines in a HQE fit, which has been improved frequently, yielding a precise determination of the HQE parameters and especially the best extracted value for $|V_{cb}|$. The HQE fit may as well serve as a test for new physics effects in these moments. The theoretical analysis of the moments reveals a low sensitivity of the moments on the non-vector currents. Thus we perform the fit with only a possible right-handed vector current contribution.

The experience in both, the HQE and the perturbative expansion enables us to consider the radiative corrections to the HQE parameters $\hat{\mu}_\pi^2$ and $\hat{\mu}_{\mathrm{G}}^2$, which are a missing puzzle piece in the precise determination of the decay $B \to X_c l\nu$. The determination of the perturbative correction to $\hat{\mu}_\pi^2$ can be easily done via reparametrization invariance, which we will present in this work. Unfortunately, the $\mathcal{O}(\alpha_s)$ corrections to $\hat{\mu}_{\mathrm{G}}^2$ can only be obtained by a full calculation. We will present briefly the strategy for such a calculation and give the result for a certain moment of the partonic invariant mass, which can be obtained by only the real corrections, which have been done.

Zusammenfassung

Eine Extraktion des CKM-Matrixelements $|V_{cb}|$ aus dem inklusiven Zerfall $B \to X_c l\nu$ liefert den genauesten Wert mit einem relativen Fehler von ca. 2 %. Dies liegt an der präzisen theoretischen Beschreibung durch die Heavy-Quark-Expansion (HQE) und den enormen Datenmengen, die von den B-Fabriken Babar und Belle gesammelt worden sind. Außer der präzisen Bestimmung von $|V_{cb}|$ und den HQE-Parametern ermöglicht uns der Zerfall auch einen Test auf Neue Physik. Normalerweise werden Zerfälle mit einem kleinen Standardmodell-Beitrag auf der Suche nach Neuer Physik untersucht. Dazu zählen insbesondere Flavour-ändernde neutrale Ströme, die durch den GIM-Mechanismus unterdrückt sind. Aber da rechtshändige schwache Ströme im Standardmodell nicht auftauchen wäre ein entsprechendes Signal ein eindeutiger Hinweis auf Neue Physik in diesem Zerfall.

Wir werden eine solche Analyse durchführen, die aus dem leptonischen Sektor als "'Michel-Parameter-Analyse"' bekannt ist. Beiträge möglicher Neuer Physik werden mit Methoden effektiver Theorien hergeleitet, die einen erweiterten $b \to c$ Strom ergeben, der nicht nur einen rechtshändigen Vektor-Beitrag hat, sondern auch skalare und tensorielle Kopplungen. Wir werden die Berechnung des Zerfalls mit dem erweiterten Strom bis $\mathcal{O}(1/m_\mathrm{b}^2)$ in der HQE und $\mathcal{O}(\alpha_s)$ in der perturbativen Entwicklung wiederholen.

Eine Momentenanalyse mit den Momenten des Leptonenergiespektrums und dem Spektrums der hadronisch invarianten Masse ist sowohl in der theoretischen Berechnung als auch in der experimentellen Bestimmung zu einem verlässlichen Werkzeug geworden. Beides fließt in den HQE-Fit ein, der regelmäßig verbessert wird und eine präzise Bestimmung der HQE-Parameter liefert, speziell den genauesten Wert für $|V_{cb}|$. Der HQE-Fit kann darüberhinaus auch als Test auf Neue Physik in diesen Momenten dienen. Die theoretische Auswertung der Momente lässt eine niedrige Sensitivität der Momente auf Nicht-Vektorströme erkennen. Daher werden wir den Fit lediglich mit einem möglichen rechtshändigen Vektorstrom durchführen.

Die Erfahrung in der HQE und der perturbativen Entwicklung ermöglicht es uns über die Strahlungskorrekturen zu den HQE-Parametern $\hat{\mu}_\pi^2$ und $\hat{\mu}_\mathrm{G}^2$ nachzudenken, die ein fehlendes Puzzleteil in der präzisen Bestimmung des Zerfalls $B \to X_c l \nu$ darstellen. Die Strahlungskorrekturen zu $\hat{\mu}_\pi^2$ können sehr einfach mit Reparametrisierungsinvarianz bestimmt werden, wie wir in dieser Arbeit zeigen werden. Leider können die Strahlungskorrekturen zu $\hat{\mu}_\mathrm{G}^2$ nur durch eine volle Rechnung ermittelt werden. Wir werden kurz die Strategie für eine solche Rechnung erläutern und das Ergebnis für ein bestimmtes Moment der partonisch invarianten Masse angeben, das nur durch die Berechnung der reellen Strahlungskorrekturen erhalten werden kann.

Contents

1 Introduction

The idea of approaching nature's laws by the use of symmetries is very old, even though its notion has changed over the years. The word *symmetry* is derived from the old Greek words "sun" ($\sigma\upsilon\mu$, a variation of 'sun' meaning 'with, together with') and "metron" ($\mu\acute{\epsilon}\tau\rho o\nu$, 'measure') giving "summetria", with the original meaning of *commensurability*. Soon the connotation of symmetry became more general with the meaning of "proportion relation". Thus it was a word embracing harmony and beauty, by describing the relation of parts to each other and to the whole. In Plato's Timaios he assigns regular polyhedrons to the natural elements at that time: the tetrahedron for fire, the octahedron for air and the icosahedron for water, the cube for earth and the dodecahedron for the aether. All of these objects are made up from triangles, which he assumed to be the fundamental pieces of matter. The elements can thus split up and rearrange their triangles to give other polyhedrons, i. e. elements. Kepler reused these "Platonic solids" in his *Mysterium Cosmographicum* published in 1596, to relate the orbits of the five planets known at that time to great circles on spheres enclosing a corresponding polyhedron. Finally, he failed to describe the proportions and the orbits in this way, and rejected his idea, by finding the orbits to be ellipses, leading to Kepler's laws.

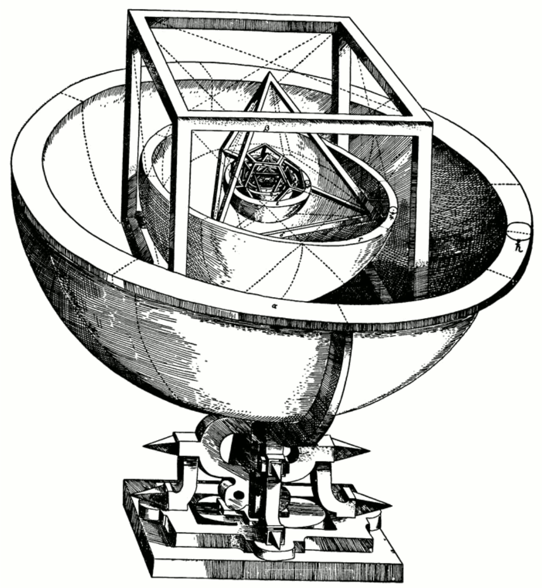


Figure 1.1: *Kepler's solar system*

The notion of "symmetry" changed and got today's meaning by Lagrange in the context of the formulation of theoretical mechanics. He used it for describing structures or setups with invariants towards certain transformations, which became today's concept of "symmetry". These transformations can be continuous e. g. rotations, translations as continuous, or discrete, e. g. reflections, rotation by discrete angles. Thus a figure is symmetric if it has parts that can be exchanged by preserving the whole. The platonic solids are thus also symmetric in this new understanding. E. g. the cube is invariant under a rotation of 90° around any axis going perpendicular through the center of two faces.

It became a common practice to study physical theories under there transformation properties, such as Poisson's brackets. In turn one could start with imposing a symmetry and derive the dynamical equations having this symmetry. This is a way of top-down approach that has widely been used in particle physics with great success, which we will see in the following chapter.

A famous example is the theory of relativity by Albert Einstein, by postulating the physical laws to be invariant under Lorentz transformations (rotations and boosts), obeying the principle of relativity and leaving the speed of light constant in all frames of references. The Lorentz symmetry was then generalized to the Poincaré symmetry, by adding translations. Every theory is assumed to be invariant under these space-time symmetry transformations.

Emmy Noether found in 1918 that an imposed symmetry yields a conserved quantity, which become known as the important *Noether's theorem*. Thus the most important conservation laws in mechanics, the conservation of energy, momentum and angular momentum were traced back to the invariance of corresponding laws under time shifts, spatial translations and rotation, respectively.

In the context of quantum mechanics the discussion of the discrete symmetries *parity* (P) (spatial reflection), *time reversal* (T) and a new particle-antiparticle symmetry *charge conjugation* (C) became important. It was shown by C. S. Wu that the weak interaction violates parity (P), even maximally, while the electromagnetism, the strong interaction and gravity are invariant with respect to C, P and T independently. The CPT theorem states that a combination of C, P and T has to be a symmetry of all physical laws. It was first believed that the combined CP transformation would be conserved by the weak interaction, which turned out to be wrong: Even CP is violated slightly. The origin and size of the CP violation is a large research field today.

The mathematical tools for describing symmetries is the field of *group theory*. Especially the description of continuous symmetries had been worked out by Sophus Lie, why they are called Lie groups. He traced a symmetry transformation back to the study of so-called generators of the corresponding infinitesimal transformation. These generator obey commutator relations forming a so-called *Lie algebra*.

Gauge theories in particle physics impose *local* and *internal* continuous symmetries on particle fields. The term "gauge" originally refers to the freedom of phase redefinitions of particle wave functions, which are not the observables of the theory. A gauge symmetry requires the Lagrangian to be invariant under the corresponding gauge transformation. By dropping the requirement to choose a phase globally for a particle wave function, its "gauge" depends on the space-time point. The requirement for the Lagrangian to be invariant under a gauge transformation (phase transformation) gives rise to an additional field, the gauge field, generating an interaction of the field at different space-time points. The prototype of such a *gauge theory* is the quantum electrodynamics (QED), which can be described by a U(1) symmetry. The principle can be generalized to more complicated internal symmetries, that involve not only phase redefinitions, but also rotations in abstract, internal spaces. This is used in the formulation of the electroweak theory, the unification of electromagnetism and weak interaction, and quantum chromodynamics, the theory of strong interactions.

Gauge theories originated from approximated symmetries, due to slight mass differences, like the strong isospin symmetry of proton and neutron. Mass terms in the Lagrangian break the imposed symmetry explicitly, which was unsatisfying as a fundamental principle. Also a gauge theory of the weak interaction yielded massless gauge bosons, which was not in accordance with experiment. *Spontaneous Symmetry Breaking* and the *Higgs mechanism* were the key to resolve these problems. Starting with massless particles and thus exact symmetries of the Lagrangian, mass terms for the particles appear through interaction terms with the so-called Higgs field, which has a degenerated lowest energy state (vacuum state). Choosing a specific vacuum expectation value yields a spontaneous breakdown of the original Lagrangian's symmetry. The Lagrangian still has this symmetry, but the lowest energy state of the Higgs field does not. On the one hand gauge theories as origin of interactions due to symmetries are thus justified as being fundamental and on the other hand mass is traced back to the interaction of particles with the Higgs field. This does not diminish the number of parameters in the standard model, because in place of a mass comes a coupling constant of the corresponding particle to the Higgs field.

Nevertheless, starting with a set of massless, non-interacting fermions and imposing certain local symmetries ends up in a precise description of all phenomenologically correct interactions of the particles. The beauty and elegance on the one side and the precise description of matter and its interactions by gauge theories on the other side is stunning. Modern particle physics has thus rekindled the old perception of symmetry as harmony, beauty and evidence for the divine origin of nature.

2 The Standard Model

2.1 Interactions

The standard model of particle physics is a relativistic quantum field theory assumed to describe matter and its interaction in the universe. It predicts 12 fundamental particles with spin 1/2 and describes the interactions between them by imposing local gauge symmetries which generates gauge bosons compensating for the change of gauge along space-time. The latter method of constructing interactions are characteristic for gauge theories. The standard model gives a uniform description of the basic interactions: the electromagnetic, the weak and the strong interaction. Unfortunately there is no promising approach to include gravitation in this model. The standard model is a gauge theory with the gauge group

$$\mathrm{SU}(3)_C \times \mathrm{SU}(2)_W \times U(1)_Y. \tag{2.1}$$

The electromagnetic and the weak interaction are united as the electroweak interaction with the gauge group $\mathrm{SU}(2)_W \times U(1)_Y$ predicting both interactions to be two aspects of one force, which can be seen at high energies, the unification scale, where electromagnetic and weak interactions are of equal strength. Quantum chromodynamics, the theory of the strong interaction, adds a further symmetry to the standard model: an exact symmetry of three colors of the six quarks, resulting in its gauge group $\mathrm{SU}(3)_C$. The standard model of particle physics is the bundle of the electroweak theory and quantum chromodynamics, but the color interaction does not unite with the electroweak interaction like electromagnetism and the weak force. Because its description is in the same terms as the electroweak theory, i. e. as gauge theory, they are legitimately packetized together.

<table>
<tr><th>interaction</th><th>field quanta</th><th>mass</th><th>strength</th><th>theory</th><th></th><th></th></tr>
<tr><td>strong</td><td>8 gluons</td><td>0</td><td>1</td><td>QCD</td><td></td><td rowspan="4">SM</td></tr>
<tr><td>electromagnetic</td><td>photon</td><td>0</td><td>10^{-3}</td><td>QED</td><td rowspan="3">GWS</td></tr>
<tr><td rowspan="2">weak</td><td>$W^{\pm}$ boson</td><td>80.43 GeV</td><td rowspan="2">10^{-14}</td><td rowspan="2"></td></tr>
<tr><td>Z^0 boson</td><td>91.19 GeV</td></tr>
<tr><td>gravitation</td><td>graviton</td><td>0</td><td>10^{-43}</td><td>GRT</td><td></td><td></td></tr>
</table>

Table 2.1: *Fundamental interactions*
(QED: quantum electrodynamics, GWS: Glashow-Weinberg-Salam theory, i. e. electroweak theory,QCD: Quantum chromodynamics, SM: standard model, GRT: general relativity; the strength is relative to the strong interaction)

	family			el. charge
	I	II	III	(e)
leptons	$\begin{pmatrix} \nu_e \\ e \end{pmatrix}_L$	$\begin{pmatrix} \nu_\mu \\ \mu \end{pmatrix}_L$	$\begin{pmatrix} \nu_\tau \\ \tau \end{pmatrix}_L$	0 -1
quarks	$\begin{pmatrix} \mathrm{u} \\ \mathrm{d} \end{pmatrix}_L$	$\begin{pmatrix} \mathrm{c} \\ \mathrm{s} \end{pmatrix}_L$	$\begin{pmatrix} \mathrm{t} \\ \mathrm{b} \end{pmatrix}_L$	2/3 -1/3

Table 2.2: *Fermions in the standard model*

2.1.1 Particle Content

Table 2.2 shows the fundamental spin 1/2 particles (fermions) in the standard model of particle physics: electron (e), muon (μ), tau (τ), electron-neutrino (ν_e), muon-neutrino (ν_μ), tau-neutrino (ν_τ), up-quark (u), down-quark (d), charm-quark (c), strange-quark (s), top-quark (t) and bottom-quark (b). They are grouped on the one hand by their participation in strong interaction: leptons have no color charge and thus do not interact strongly transforming as a singlet under $\mathrm{SU}(3)_\mathrm{C}$. Quarks have three possible color charges transforming as a triplet under $\mathrm{SU}(3)_\mathrm{C}$. On the other hand the fermions are grouped into so called families, each with a charged lepton, a neutrino, and two quarks of the charges 2/3 and $-1/3$. Except for the different masses the corresponding particles of each family have equal properties. The type of fermion is also called "flavour", depending on the sector either "quark flavour" or "lepton flavour". The notion of flavour and the origin of the three families is still a mystery. The families are implemented in the theory as triplication of the first family, which was first being discovered, because matter is made of the particles therein. The following introduction follows [1].

The particles in table 2.2 are arranged in doublets indicating their properties referring to weak interactions. The historically origin is the concept of isospin-invariance of the strong interaction introduced by Heisenberg. As proton and neutron have almost equal mass he proposed to consider them as different states of one particle named "nucleon" with the strong interaction unable to distinguish. This was done in reference to the spin of a particle yielding also the name "isospin". The proton and neutron wave function can then be written as the product of ψ_N of the nucleon and a so-called isospinor χ:

$$\chi_p = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad \text{and} \qquad \chi_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{2.2}$$

Tracing back the mass difference to the different electromagnetic interactions of proton and neutron, neglecting these interactions led to a SU(2) symmetry with the proton and neutron being two different occurrences of the nucleon.

A symmetry transformation U can e. g. turn a neutron into a proton as in the neutron decay

$$n \longrightarrow p\, e\, \nu_e.$$

The transformation U has to be unitary ($UU^\dagger = 1$) to conserve the normalization. Due to $\det(UU^\dagger) = |\det U|^2 = 1$ the transformation can be written as $e^{i\alpha}$.

The SU(2) is a continuous group, also known as *Lie group.* All group elements of a Lie group can be constructed by a finite group transformation from the unity element with the means of so-called *generators.* The generators are crucial in the discussion of a Lie group, because they reflect its properties. The generators form a so-called *Lie algebra* and describe directly an infinitesimal group transformation. Starting from an infinitesimal transformation

$$U = 1 + \mathrm{i}\xi$$

we can construct any finite group transformation. Because of the transformation being unitary $1 = UU^\dagger = (1 - \mathrm{i}\xi^\dagger)(1 + \mathrm{i}\xi) \approx 1 + \mathrm{i}(\xi - \xi^\dagger)$ the infinitesimal transformation ξ has to be hermitian ($\xi = \xi^\dagger$). Also $\operatorname{Tr}\xi = 0$ due to $\det(U) = 1$. The presented doublets indicate that a representation of the SU(2) with 2×2 matrices exists. An orthogonal set of hermitian and traceless 2×2 matrices are the Pauli matrices

$$\tau_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \tau_2 = \begin{pmatrix} 0 & -\mathrm{i} \\ \mathrm{i} & 0 \end{pmatrix} \qquad \tau_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.3}$$

which we label by τ_1, τ_2 and τ_3 for the use in the context of isospin, to distinguish them from the Pauli spin matrices σ_1, σ_2 and σ_3. An infinitesimal transformation can be written with $\xi = \frac{1}{2}(\epsilon_i \tau_i)$ as

$$U = 1 + \frac{\mathrm{i}}{2}(\epsilon_i \tau_i).$$

This can be compared to a three dimensional rotation (SO(3)), not least owing to the fact that the SU(2) is isomorphic to the SO(3), the ϵ_i describe "three rotation angles" in isospin space.

Any finite transformation α can be constructed from subsequent infinitesimal transformations. For a sufficient big n the transformation $\epsilon = \alpha/n$ becomes infinitesimal and we can write the finite transformation as

$$U = \lim_{n\to\infty} \left(1 + \frac{\mathrm{i}}{2}\frac{\alpha_i \tau_i}{n}\right)^n = e^{\frac{\mathrm{i}}{2}\alpha_i \tau_i}. \tag{2.4}$$

In the last step we used the definition of the exponential function. Redefining the generators to

$$T_i = \frac{1}{2}\tau_i$$

with $i = 1, 2, 3$ we find a simple Lie algebra as commutator relations:

$$[T_i, T_j] = \mathrm{i}\epsilon_{ijk} T_k \tag{2.5}$$

The third generator T_3 has the nice feature, that its eigenvalues display the nucleon state. The value $+1/2$ for the proton and $-1/2$ for the neutron. The charge operator is then

$$Q = \mathrm{e}\left(T_3 + \frac{1}{2}\right) \tag{2.6}$$

giving the right electrical charges: +e for the proton and 0 for the neutron.

Parity Violation of the Weak Interaction

Phenomenologically only the left- handed part of a particle field is subjected to the weak interaction. Therefore, we decompose the fermion fields ψ into their left-handed and right-handed parts (ψ_L and ψ_R):

$$\psi = \psi_L + \psi_R \qquad \text{with} \qquad \psi_L = \frac{1}{2}(1 - \gamma^5)\psi \quad \text{and} \quad \psi_R = \frac{1}{2}(1 + \gamma^5)\psi$$

and assign different weak quantum numbers to them. The left-handed particles are assigned to SU(2) doublets, as indicated in 2.2, by the subscript L, and the right-handed to the corresponding singlets:

$$e_R, \mu_R, \tau_R, u_R, d_R, c_R, s_R, t_R \quad \text{and} \quad b_R.$$

2.2 The Gauge Theory of Electroweak Interactions

The SU(2) symmetry in conjunction with the U(1) symmetry from electromagnetism assumed to be fundamental and applied as "weak isospin" to the elementary particles, i. e. the quarks and lepton as the theory of electroweak interactions by Glashow, Weinberg and Salam (GWS theory), which we will describe in this section. Big mass differences don't hamper its application, because the particles acquire their mass by interactions with the so-called Higgs field and the Higgs mechanism describing a spontaneous symmetry breaking. The unbroken Lagrangian respects the SU(2) × U(1) symmetry.

The electroweak theory combines the weak interaction with the electromagnetic interaction by assuming a *local* $\mathrm{SU}(2)_L \times \mathrm{U}(1)$ symmetry yielding four gauge bosons, one for every generator of the symmetry groups, three for the SU(2) and one for the $U(1)$. A first result does not yield the right phenomenology, i. e. the gauge boson field of the $U(1)$ couples to neutrinos, even though they have no electrical charge. A special linear combination of the third gauge boson from the $\mathrm{SU}(2)_L$ symmetry with the boson from the U(1) can be adjusted to a vanishing coupling to the neutrinos, yielding the right phenomenological electromagnetic field. We will describe this derivation in detail in the following.

The symmetry group of the electroweak interaction is

$$\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y.$$

As the $\mathrm{SU}(2)_L$ in the discussion of the strong interaction above, the $\mathrm{SU}(2)_W$ of the weak interaction has three generators τ_1, τ_2 and τ_3 and three parameters describing the transformation of left-handed doublets. For the case of the doublet with electron and electron-neutrino the transformation can be written as

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \to \mathrm{e}^{\mathrm{i}gT_a\beta_a(x)} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L . \tag{2.7}$$

with $T_a = \tau_a/2$. Right-handed particle fields transform as singlets under $\mathrm{SU}(2)_W$, e. g. $e_R \to e_R$. The group $U(1)_Y$ has only one generator, the *hypercharge* Y, yielding a phase transformation for the left-handed $\mathrm{SU}(2)_W$ doublets, like

$$\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L \to \mathrm{e}^{\mathrm{i}\frac{g'}{2}Y_L\xi(x)} \begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L . \tag{2.8}$$

and for the right-handed $\mathrm{SU}(2)_W$ singlets:

$$e_R \to \mathrm{e}^{\mathrm{i}\frac{g'}{2}Y_R\xi(x)}\, e_R. \tag{2.9}$$

The hypercharge group $\mathrm{U}(1)_Y$ has to be included, as a weak interacting neutral boson is needed to cancel certain divergences in the W boson model of the weak interaction. We will see that this requirement leads to an inclusion of the electromagnetic interaction into one theory. Using the $U(1)$ group of electromagnetism and the electrical charge as a generator would not be a valid transformation of the left-handed doublet, because the charge of e. g. the electron and the neutrino are different. A common quantum number for the doublet is the weak hypercharge Y, named in analogy to the charge, as a combination of the electrical charge and the third weak isospin component I_3:

$$Y = 2(Q - I_3). \tag{2.10}$$

particle type	family			Q	I3	Y
left-handed leptons	$\begin{pmatrix}\nu_e \\ e^-\end{pmatrix}_L$	$\begin{pmatrix}\nu_\mu \\ \mu^-\end{pmatrix}_L$	$\begin{pmatrix}\nu_\tau \\ \tau^-\end{pmatrix}_L$	0 -1	1/2 -1/2	-1
right-handed leptons	e_R^-	μ_R^-	τ_R^-	-1	0	-2
left-handed quarks	$\begin{pmatrix}u \\ d\end{pmatrix}_L$	$\begin{pmatrix}c \\ s\end{pmatrix}_L$	$\begin{pmatrix}t \\ b\end{pmatrix}_L$	2/3 -1/3	1/2 -1/2	1/3
right-handed up-type quarks	u_R	c_R	t_R	2/3	0	4/3
right-handed down-type quarks	d_R	s_R	b_R	-1/3	0	-2/3

Table 2.3: *Electrical charge (Q), weak isospin (I_3) and hypercharge (Y) of fundamental fermions*

All symmetries are *local* gauge symmetries, i. e. the transformations depend on the space-time point x. Symmetry implies the invariance of the Lagrangian of symmetry transformations. In the case of a local symmetry gauge fields compensate for the extra terms from derivatives of the gauge transformation. Each generator is accompanied by a transformation parameter depending on space-time, thus we need as many gauge fields as group generators. For the $\mathrm{SU}(2)_W$ we need three vector fields W_1^μ, W_2^μ and W_3^μ and for the $\mathrm{U}(1)_Y$ only one vector field B^μ. The compensating gauge fields are introduced in the theory by extending the derivative to a covariant derivative:

$$D_\mu = \partial_\mu + \mathrm{i} g T^a W_\mu^a + \mathrm{i}\frac{g'}{2} Y B_\mu. \tag{2.11}$$

which compensated for the symmetry transformations of the $\mathrm{SU}(2)_W$ (2.7) and $\mathrm{U}(1)_Y$ (2.8) if the fields W_μ^a and B_μ transform according to

$$W_\mu^a \to W_\mu^a + \frac{1}{g}\partial_\mu \beta^a + \epsilon^{abc} W_\mu^b \beta^c \tag{2.12}$$

$$B_\mu \to B_\mu + \frac{1}{g'}\partial_\mu \chi \tag{2.13}$$

The concept of covariant derivatives is also known from general relativity to compensate for the change of the coordinate system. A more related application is the coupling to the electromagnetic field in quantum electrodynamics known as *minimal coupling.* For left-handed fermions it is $T^a = \tau^a/2$, with the eigenvalues I_3. Thus we have for left-handed leptons (with $Y = -1$)

$$D_\mu = \partial_\mu + \mathrm{i}\frac{g}{2}\tau^a W_\mu^a - \mathrm{i}\frac{g'}{2} B_\mu$$

and for right handed leptons ($T^a = 0$ and $Y = -2$):

$$D_\mu = \partial_\mu - \mathrm{i} g' B_\mu.$$

Considering the possible symmetry transformations we find that transitions like $e^- \to \nu_e$ can be mediated by the isospin raising operator τ_+ under emission of a W^- boson and the reverse transition $\nu_e \to e^-$ by the isospin lowering operator τ_- under emission of a W^+ boson:

$$\tau_+ = \frac{1}{2}(\tau_1 + \mathrm{i}\tau_2) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad \tau_- = \frac{1}{2}(\tau_1 - \mathrm{i}\tau_2) = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{2.14}$$

If we construct the physical $W^\pm$ boson fields as charge eigenstates

$$W^\pm_\mu = \frac{1}{\sqrt{2}} \left(W_{1,\mu} \pm W_{2,\mu} \right),$$

we can rearrange the covariant derivative for the left-handed leptons accounting for these fields:

$$D_\mu = \partial_\mu + \mathrm{i}\frac{g}{\sqrt{2}} \left(\tau_+ W^-_\mu + \tau_- W^+_\mu \right) + \mathrm{i}\frac{g}{2}\tau_3 W_{3,\mu} - \mathrm{i}\frac{g'}{2} B_\mu. \tag{2.15}$$

The last two terms $\mathrm{i}\frac{g}{2}\tau_3 W_{3,\mu}$ and $-\mathrm{i}\frac{g'}{2} B_\mu$ mediate both individually a neutral current of the neutrinos, e. g. $\nu_e \to \nu_e$. None of them can be identified with the electromagnetic field A_μ, because this doesn't couple to the chargeless neutrinos. But we can choose a linear combination is such a way that the coupling to the neutrino vanishes:

$$A_\mu = a W_{3,\mu} + b B_\mu.$$

The coupling of the neutrinos is proportional to

$$a \left(-\frac{g}{2} \right) + b\frac{g'}{2} = 0,$$

which yields $a = bg'/g$. Normalizing a and b according to $\sqrt{a^2 + b^2} = 1$ leads to

$$a = \frac{g'}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad b = \frac{g}{\sqrt{g^2 + g'^2}}. \tag{2.16}$$

The normalization in form of the Pythagorean theorem in the unit circle suggests to write a and b as

$$\sin\theta_\mathrm{W} = \frac{g'}{\sqrt{g^2 + g'^2}} \quad \text{and} \quad \cos\theta_\mathrm{W} = \frac{g}{\sqrt{g^2 + g'^2}} \tag{2.17}$$

with the so-called *Weinberg angle* θ_W. This yields a mixing relation for A_μ of $W_{3,\mu}$ and B_μ, but gives also rise to the orthogonal mixing to a neutral weak field Z_μ, related to the neutral weak boson Z^0:

$$\begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{W} & -\sin\theta_\mathrm{W} \\ \sin\theta_\mathrm{W} & \cos\theta_\mathrm{W} \end{pmatrix} \begin{pmatrix} W_{3,\mu} \\ B_\mu \end{pmatrix} \tag{2.18}$$

Solving (2.18) to the initial fields

$$\begin{pmatrix} W_{3,\mu} \\ B_\mu \end{pmatrix} = \begin{pmatrix} \cos\theta_\mathrm{W} & \sin\theta_\mathrm{W} \\ -\sin\theta_\mathrm{W} & \cos\theta_\mathrm{W} \end{pmatrix} \begin{pmatrix} Z_\mu \\ A_\mu \end{pmatrix} \tag{2.19}$$

allows us to rewrite the covariant derivative in terms of A_μ and Z_μ:

$$D_\mu = \partial_\mu + \mathrm{i}\frac{g}{\sqrt{2}}\left(\tau_+ W_\mu^- + \tau_- W_\mu^+\right) + \frac{\mathrm{i}}{2}\left(g\cos\theta_\mathrm{W}\tau_3 + g'\sin\theta_\mathrm{W}\right) Z_\mu + \frac{\mathrm{i}}{2}\left(g\sin\theta_\mathrm{W}\tau_3 - g'\cos\theta_\mathrm{W}\right) A_\mu. \tag{2.20}$$

Due to $\nu_L = u_L \begin{pmatrix}1\\0\end{pmatrix}$ it is $\tau_3\nu_L = \nu_L$ an the coupling of the gauge bosons to the neutrinos is

$$\frac{\mathrm{i}}{2}\underbrace{\left(g\sin\theta_\mathrm{W} - g'\cos\theta_\mathrm{W}\right)}_{0}\bar{u}_L\gamma^\mu u_L A_\mu + \frac{\mathrm{i}}{2}\underbrace{\left(g\cos\theta_\mathrm{W} + g'\sin\theta_\mathrm{W}\right)}_{g/\cos\theta_\mathrm{W}}\bar{u}_L\gamma^\mu u_L Z_\mu = \frac{\mathrm{i}g}{2\cos\theta_\mathrm{W}}\bar{u}_L\gamma^\mu u_L Z_\mu.$$

The obtained electromagnetic field A_μ does not couple to the neutrino as demanded, but to the neutral weak field Z_μ. The existence of a neutral weak boson was originally required to cancel divergences in certain weak decays and appears here in a natural way as an orthogonal field combination of the electromagnetic field, demonstrating the unique source of the weak and electromagnetic force. Considering the coupling of the electron we can for simplicity refer to the right-handed part, which does not couple to $W_{3,\mu}$:

$$\mathrm{i}g'\bar{u}_R\gamma^\mu u_R B_\mu = \mathrm{i}g'\cos\theta_\mathrm{W}\bar{u}_R\gamma^\mu u_R A_\mu - \mathrm{i}g'\sin\theta_\mathrm{W}\bar{u}_R\gamma^\mu u_R Z_\mu. \tag{2.21}$$

In QED the coupling of the photon in independent from the handedness:

$$\mathrm{i}\,\mathrm{e}\,\bar{u}\gamma^\mu u A_\mu$$

A coefficient comparison with (2.21) gives the relation between the elementary charge and the coupling constants:

$$\mathrm{e} = g'\cos\theta_\mathrm{W} = g\sin\theta_\mathrm{W}. \tag{2.22}$$

Unfortunately, the weak bosons $W^\pm$ and Z^0 are massless, contrary to observation. But a related mass term would break gauge invariance. The Higgs mechanism discussed in the next section provides a remedy, giving mass to gauge bosons and also to fermions in a gauge invariant way by means of spontaneous symmetry breaking.

2.2.1 Higgs Mechanism

The Higgs mechanism allows to trace back mass terms to couplings with the non-vanishing vacuum expectation value of a so-called Higgs field. Mass terms for gauge bosons and fermions are not included in the Lagrangian from the start maintaining gauge invariance. The Higgs field is introduced with a potential leading to a non-vanishing vacuum expectation value. The covariant derivatives in the kinetic term for the Higgs field yield the interaction with the gauge bosons. Interactions with fermions can be constructed in a gauge invariant way via so-called Yukawa couplings. A spontaneous symmetry breaking in form of going to the vacuum expectation value of the Higgs field yields mass terms for gauge bosons and fermions coding the mass by the vacuum expectation value of the Higgs field and the coupling of the particle to the Higgs field. This does not reduce the number of parameters in the theory, because masses turn to couplings to the Higgs field, but it explains the origin of mass and introduces mass in a gauge invariant way. All assumed symmetries are thus exact and only broken spontaneously.

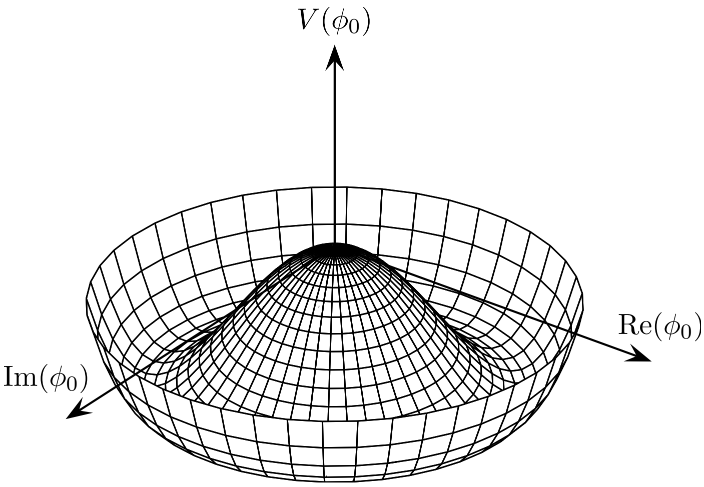


Figure 2.1: *Higgs potential of the field* ϕ^0

We start with a Higgs $SU(2)_W$ doublet with hypercharge $Y=1$ consisting of two complex fields ϕ_+ and ϕ_0:

$$\Phi = \begin{pmatrix} \phi_+ \\ \phi_0 \end{pmatrix}. \tag{2.23}$$

with ϕ_+ describing a charged field and ϕ_0 a neutral field. This is the simplest possible Higgs structure to produce mass terms for the $W^{\pm}$ and Z^0 bosons. An occurring Higgs boson has not been found yet. Thus the Higgs structure realized in nature is not clear. E. g. a two-Higgs-doublet model is discussed (see e. g. [2]).

The part of the Lagrangian describing the Higgs field

$$\mathscr{L}_{\mathrm{Higgs}} = (\mathrm{D}^{\mu}\,\Phi)(\mathrm{D}_{\mu}\,\Phi) - V(\Phi^{\dagger}\Phi) \tag{2.24}$$

contains a potential

$$V(\Phi^{\dagger}\Phi) = -\mu^2\Phi^{\dagger}\Phi + \lambda^2(\Phi^{\dagger}\Phi)^2 \tag{2.25}$$

with $\mu, \lambda > 0$, that yields a non-trivial minimum for $|\phi_0|$, if we assume $|\phi_+|$ to be identical to zero starting from

$$-\mu^2\,|\phi_0|^2 + \lambda^2\,|\phi_0|^4$$

and determining the minimum via derivation with respect to $|\phi_0|$ and equating to zero, yielding

$$|\phi_0| = \frac{1}{\sqrt{2}}\frac{\mu}{\lambda} = \frac{v}{\sqrt{2}} \qquad \text{with} \qquad \text{with} \quad v = \frac{\mu}{\lambda}.$$

for the absolute value of $|\phi_0|$ (besides $|\phi_0| = 0$, which turns out to be a maximum).

The Higgs Lagrangian is invariant under gauge transformations of the Higgs field Φ, assured by the use of the covariant derivative. This is an important property, because we want to obtain mass terms for bosons and fermions in a gauge invariant way. Adding a field explicitly breaking this invariance would be counterproductive.

The assumption of the Higgs mechanism is, that the lowest possible state, the vacuum state, takes a non-vanishing value $v/\sqrt{2}$ for the neutral Higgs field, assured by the Higgs potential, while the charged field is zero, which otherwise would lead to a photon mass:

$$\langle 0|\Phi|0\rangle = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v\end{pmatrix} \tag{2.26}$$

A state in the vicinity of the vacuum expectation value $\Phi(x)$ is

$$\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ v+\eta(x)+i\zeta(x)\end{pmatrix}. \tag{2.27}$$

The field $\zeta(x)$ describes a massless particle, the so-called *Goldstone boson*. It is an element of Goldstone's theorem stating that a massless particle appears if the Lagrangian has an exact continuous symmetry which the ground state doesn't share. It can be eliminated by a $U(1)_Y$ gauge transformation becoming a longitudinal mode of the massive gauge bosons, and thus we will drop it directly.

The application of the covariant derivative yields

$$\mathrm{D}_\mu\Phi(x) = \frac{1}{\sqrt{2}}\begin{pmatrix}0\\ \partial_\mu\eta\end{pmatrix} + \frac{\mathrm{i}g(v+\eta)}{2}\begin{pmatrix}W_\mu^-\\ 0\end{pmatrix} - \frac{\mathrm{i}g(v+\eta)}{2\sqrt{2}\cos\theta_\mathrm{W}}\begin{pmatrix}0\\ Z_\mu\end{pmatrix}. \tag{2.28}$$

(Please note that the hypercharge of the Higgs doublet is $Y=1$ in contrast to $Y=-1$ for the electron, which we used for the derivation for (2.20). Thus the signs in front of all terms involving g' have to be switched.) Aiming at the kinetic term for the Higgs doublet we multiply the hermitian conjugate from the left: (note that $(W_\mu^-)^\dagger = W_\mu^+$):

$$(\mathrm{D}^\mu\Phi)^\dagger(\mathrm{D}_\mu\Phi) = \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) + \frac{g^2(v+\eta)^2}{4}\left|W_\mu^-\right|^2 + \frac{g^2(v+\eta)^2}{8\cos^2\theta_\mathrm{W}}\left|Z_\mu\right|^2. \tag{2.29}$$

With

$$|W_\mu^-|^2 = (W^{-\mu})^\dagger W_\mu^- = \frac{1}{2}(W_1^\mu + \mathrm{i}W_2^\mu)(W_{1,\mu} - \mathrm{i}W_{2,\mu}) = W_\mu^+(W^{+\mu})^\dagger = |W_\mu^+|^2 = \frac{1}{2}\left(|W_\mu^+|^2 + |W_\mu^-|^2\right)$$

and the approximation $v+\eta \approx v$ we plug the result in the Lagrangian (2.24) and apply the vacuum expectation value to the potential (2.25) with $\lambda = \mu/v$, neglecting constant terms $(-1/4\mu^2v^2)$ and terms of $\mathcal{O}(\eta^3)$:

$$\mathscr{L}_\mathrm{Higgs} = \frac{1}{2}(\partial^\mu\eta)(\partial_\mu\eta) - \mu^2\eta^2 + \frac{1}{2}\frac{g^2v^2}{4}\left(|W_\mu^+|^2 + |W_\mu^-|^2\right) + \frac{1}{2}\frac{g^2v^2}{4\cos^2\theta_\mathrm{W}}\left|Z_\mu\right|^2. \tag{2.30}$$

From the equation (2.30) we can read of:

- A neutral Higgs particle, the η particle, appears with the masse

$$M_\mathrm{Higgs} = \sqrt{2}\mu. \tag{2.31}$$

- The charged W bosons acquire a mass of

$$M_{W^\pm} = \frac{gv}{2}. \tag{2.32}$$

- The neutral Z boson acquire a mass of

$$M_{Z^0} = \frac{gv}{2\cos\theta_\mathrm{W}} = \frac{M_{W^\pm}}{\cos\theta_\mathrm{W}}, \tag{2.33}$$

presenting an important relationship between the charged $W^\pm$ boson mass and the neutral boson Z^0 mass, which has been tested experimentally to validate the theory of weak interaction.

- The photon does not acquire a mass, which already becomes obvious by the absence of A_μ in (2.28), as required by observation.

Fermion Mass Terms The introduction of the Higgs doublet amounts for interaction terms with fermion fields. It is possible to couple left- and right-handed fermion field components via the Higgs field of the form

$$\overline{L}_L \Phi e_R^- + \text{h.c.} = \begin{pmatrix} \overline{\nu}_{e,L} & \overline{e}_L^- \end{pmatrix} \begin{pmatrix} \phi^+ \\ \phi^0 \end{pmatrix} e_R^- + \text{h.c.}$$

which is a so-called *Yukawa coupling.* The shortcut h.c. denotes the hermitian conjugate of the term in front. Couplings of this type are gauge invariant under $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$. A coupling between left and right-handed field components point at possible mass terms for the fermions. In fact, spontaneous symmetry breaking leads to such a mass term involving the vacuum expectation value of the neutral Higgs field as well as the coupling of the fermion to the Higgs field. Introducing mass terms without Higgs interaction into the Lagrangian breaks the $\mathrm{SU}(2)_W \times \mathrm{U}(1)_Y$ symmetry, because left- and right-handed fields belong to different $\mathrm{SU}(2)_W$ representations (doublet and singlet) and they have different U(1) charges.

In this fashion we construct Yukawa couplings for all fermions:

$$\mathscr{L}_\text{Yukawa} = -g_e^{ij} \overline{L}_L^i \Phi e_R^j - g_d^{ij} \overline{Q}_L^i \Phi d_R^j - g_u^{ij} \overline{Q}_L^i \Phi^c u_R^j + \text{h.c.} \tag{2.34}$$

with

$$L_L^i = \left(\begin{pmatrix} \nu_e \\ e^- \end{pmatrix}_L , \begin{pmatrix} \nu_\mu \\ \mu^- \end{pmatrix}_L , \begin{pmatrix} \nu_\tau \\ \tau^- \end{pmatrix}_L \right) \tag{2.35}$$

$$Q_L^i = \left(\begin{pmatrix} u \\ d \end{pmatrix}_L , \begin{pmatrix} c \\ s \end{pmatrix}_L , \begin{pmatrix} t \\ b \end{pmatrix}_L \right) \tag{2.36}$$

$$e_R^j = \left(e_R^- , \mu_R^- , \tau_R^- \right) \tag{2.37}$$

$$u_R^j = \left(u_R , c_R , t_R \right) \tag{2.38}$$

$$d_R^j = \left(d_R , s_R , b_R \right) \tag{2.39}$$

The g_e^{ij}, g_d^{ij} and g_u^{ij} are Yukawa coupling matrices of the charged leptons, down-type quarks and up-type quarks respectively. Please note that for the generation of the up-type quark masses we have to use the charge conjugate of the Higgs doublet:

$$\Phi^c = i\tau_2 \Phi^* = \begin{pmatrix} \phi^0 \\ \phi^- \end{pmatrix}. \tag{2.40}$$

where the vacuum expectation value is obtained analogously to the normal Higgs doublet.

After spontaneous symmetry breaking, i. e.

$$\Phi \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} 0 \\ v+\eta \end{pmatrix} \quad \text{and} \quad \Phi^c \longrightarrow \frac{1}{\sqrt{2}}\begin{pmatrix} v+\eta \\ 0 \end{pmatrix} \tag{2.41}$$

and neglecting η, the Yukawa coupling part of the Lagrangian becomes the fermion mass part

$$\mathscr{L}_{\text{Yukawa}} = -\left(M_e^{ij}\overline{e}_L^i e_R^j + M_d^{ij}\overline{d}_L^i d_R^j + M_u^{ij}\overline{u}_L^i u_R^j + \text{h.c.}\right) \tag{2.42}$$

with $M_{e,u,d}^{ij} = \frac{v}{\sqrt{2}} g_{e,u,d}^{ij}$ and

$$e_L^j = (\ e_L^- \ , \mu_L^- \ , \tau_L^- \ ) \tag{2.43}$$

$$u_L^j = (\ u_L \ , c_L \ , t_L \ ) \tag{2.44}$$

$$d_L^j = (\ d_L \ , s_L \ , b_L \ ) \tag{2.45}$$

To obtain the regular form of the mass terms the matrices $M_{e,u,d}^{ij}$ have to be diagonal. We could apply a unitary transformation U_L and U_R to the corresponding fields that makes the matrices diagonal. This is not a problem for the charged leptons, whose fields are then transformed according to

$$e_L^i \to U_L^{ik} e_L^k \quad \text{and} \quad e_R^j \to U_R^{jl} e_R^l$$

resulting in a diagonal mass matrix:

$$M_e^{ij}\overline{e}_L^i e_R^j \to \overline{e}_L^k \underbrace{U_L^{\dagger\,ik} M_e^{ij} U_R^{jl}}_{\delta_{kl} m_l} e_R^l \tag{2.46}$$

The resulting charged lepton fields are the mass eigenstates. This is possible because we assumed the neutrinos to be massless.

In the quark sector this becomes more difficult, because the up-type quarks are not massless and we have to consider that transforming the fields affects all occurrences in the Lagrangian: The $W^\pm$ interaction term connects up-type and down-type quarks and thus a diagonalization of both mass matrices M_u^{ij} and M_d^{ij} simultaneously emerges the transformation matrices in the charged weak interactions. Instead of transforming both the up-type and down-type quarks it is also possible to define one type as mass eigenstates and transform only the other type. Conventionally the up type quarks are assumed to be in their mass eigenstates and the down-type quarks not. They are in the so-called flavour eigenstate denoted with a prime ($d_L'^i$), because they are diagonal with respect to the weak interaction, i. e. no family mixing occurs, like the decay $s' \to u$:

$$d_L^i = U_{d,L}^{ik} d_L'^k. \tag{2.47}$$

The transformation matrix from the weak eigenstates to the mass eigenstates is called the *Cabibbo-Kobayashi-Maskawa matrix* (CKM matrix):

$$V_{\text{CKM}} = U_{d,L}^{ik} \tag{2.48}$$

The CKM matrix is a unitary matrix which appears in the charged weak interaction. It is not necessary a diagonal matrix and can mediate family mixing, e. g. transitions like $s \to u$ are possible. The CKM matrix plays a crucial role in flavour physics, because it encodes all quark mixing and CP violation. Determining its elements is of major importance for the determination of CP violation, the calculation of all weak decays of quarks and especially its unitarity can be tested as a probe of the standard model.

2.2.2 The Cabibbo-Kobayashi-Maskawa Matrix

The quark mixture is mediated by a unitary 3×3 matrix involving three rotation angles and one phase.

$$\begin{pmatrix} d' \\ s' \\ b' \end{pmatrix} = \begin{pmatrix} V_{\mathrm{ud}} & V_{\mathrm{us}} & V_{\mathrm{ub}} \\ V_{\mathrm{cd}} & V_{\mathrm{cs}} & V_{\mathrm{cb}} \\ V_{\mathrm{td}} & V_{\mathrm{ts}} & V_{\mathrm{tb}} \end{pmatrix} \begin{pmatrix} d \\ s \\ b \end{pmatrix} \tag{2.49}$$

Properties of the CKM matrix To deduce the number of free parameters of the CKM matrix we consider first a general unitary $n \times n$ matrix.

A unitary $n \times n$ matrix has

- $2n^2$ real parameters: n^2 absolute values and n^2 phases, or n^2 real parts and n^2 imaginary parts.
- The unitarity relation $V^\dagger V = 1$ is a constraint with n^2 relations and thus only $2n^2 - n^2 = n^2$ independent parameters are left.

The n^2 independent parameters of a general $n \times n$ matrix reduce further for the case of the CKM matrix, because the $2n - 1$ relative phases of the n up-type and n down-type quark fields can be chosen freely. Hence the number of independent parameters is $N = n^2 - 2n + 1 = (n-1)^2$:

$$N_{\mathrm{angles}} = \frac{n(n-1)}{2} \qquad \text{and} \qquad N_{\mathrm{phases}} = \frac{(n-1)(n-2)}{2}.$$

For the case of $n = 2$, representing the perception of the GIM mechanism[1], we have only one angle (the Cabibbo angle Θ_c) an no phase.

With $n = 3$ as the CKM matrix in the standard model we have three angles and one phase. A phase is crucial for the description of the observed CP violation in the framework of Kobayashi and Maskawa. Thus we need at least three quark families to include CP violation. Two families do not suffice.

Parametrization of the CKM matrix The Standard parametrization of the Particle Data Group is a product of three rotations and a transformation with one phase. The rotations are described by three Euler like angles θ_{12}, θ_{13}, θ_{23}. As an abbreviation we use $c_{ij} = \cos\theta_{ij}$ and $s_{ij} = \sin\theta_{ij}$:

$$\begin{aligned} V_{\mathrm{CKM}} &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & -s_{23} & c_{23} \end{pmatrix} \begin{pmatrix} c_{13} & 0 & s_{13}\, e^{-i\delta} \\ 0 & 1 & 0 \\ -s_{13} e^{i\delta} & 0 & c_{13} \end{pmatrix} \begin{pmatrix} c_{12} & s_{12} & 0 \\ -s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \\ &= \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & s_{23}c_{13} \end{pmatrix} \end{aligned}$$

[1] with four quarks (up, down, charm and strange quark)

To emphasize the relative sizes of the CKM matrix elements Wolfenstein introduced an approximative parametrization based on powers of a parameter $\lambda = \sin\theta_c \approx 0{,}22$ being the sine of the Cabibbo angle:

$$V_{\text{CKM}} \approx \begin{pmatrix} 1-\lambda^2/2 & \lambda & A\lambda^3(\rho - i\eta) \\ -\lambda & 1-\lambda^2/2 & A\lambda^2 \\ A\lambda^3(1-\rho-i\eta) & -A\lambda^2 & 1 \end{pmatrix} \tag{2.50}$$

The Wolfenstein Parametrization is an approximation neglecting $\mathcal{O}(\lambda^4)$ terms. To include higher orders the modified Wolfenstein parameters $\bar\rho$ and $\bar\eta$ are used, which are at $\mathcal{O}(\lambda^5)$ [3]:

$$\bar\rho = \rho\left(1 - \frac{\lambda^2}{2}\right) \quad \text{and} \quad \bar\eta = \eta\left(1 - \frac{\lambda^2}{2}\right) \tag{2.51}$$

Hierarchy of the CKM matrix As indicated by the Wolfenstein parametrization the CKM matrix has a strong hierarchy in the absolute values of its elements. The absolute values of the matrix elements according to the Particle Data Group are

$$\begin{aligned} |V_{\text{CKM}}| &= \begin{pmatrix} |V_{\text{ud}}| & |V_{\text{us}}| & |V_{\text{ub}}| \\ |V_{\text{cd}}| & |V_{\text{cs}}| & |V_{\text{cb}}| \\ |V_{\text{td}}| & |V_{\text{ts}}| & |V_{\text{tb}}| \end{pmatrix} \\ &= \begin{pmatrix} 0{,}9745 \text{ to } 0{,}9757 & 0{,}219 \text{ to } 0{,}224 & 0{,}002 \text{ to } 0{,}005 \\ 0{,}218 \text{ to } 0{,}224 & 0{,}9736 \text{ to } 0{,}9750 & 0{,}036 \text{ to } 0{,}049 \\ 0{,}004 \text{ to } 0{,}014 & 0{,}034 \text{ to } 0{,}046 & 0{,}9989 \text{ to } 0{,}9993 \end{pmatrix} \\ &\propto \begin{pmatrix} \Huge\bullet & \Large\bullet & \cdot \\ \Large\bullet & \Huge\bullet & \small\bullet \\ \cdot & \small\bullet & \Huge\bullet \end{pmatrix} \end{aligned}$$

The area of the bullets are a rough graphical representation for the relative sizes. The matrix of the absolute values is approximately a unity matrix. The origin of this hierarchy is completely unknown and its discovery a major goal of today's research in particle physics.

Unitarity Triangle The unitarity relation for the CKM matrix can be expressed in two ways

$$V^\dagger_{\text{CKM}} V_{\text{CKM}} = V_{\text{CKM}} V^\dagger_{\text{CKM}} = 1,$$

which gives gives $2 \times 6 = 12$ possible relations of orthogonality of rows and columns:

$$V^\dagger_{\text{CKM}} V_{\text{CKM}} = \begin{pmatrix} V^*_{\text{ud}} & V^*_{\text{cd}} & V^*_{\text{td}} \\ V^*_{\text{us}} & V^*_{\text{cs}} & V^*_{\text{ts}} \\ V^*_{\text{ub}} & V^*_{\text{cb}} & V^*_{\text{tb}} \end{pmatrix} \begin{pmatrix} V_{\text{ud}} & V_{\text{us}} & V_{\text{ub}} \\ V_{\text{cd}} & V_{\text{cs}} & V_{\text{cb}} \\ V_{\text{td}} & V_{\text{ts}} & V_{\text{tb}} \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{2.52}$$

These relations can be pictured as triangles in the complex plane of the Wolfenstein parameters. Most of the triangles are of extreme shape using the known data for the parameters. But choosing a special relation indicated by the borders in (2.52)

$$V^*_{\text{ub}} V_{\text{ud}} + V^*_{\text{cb}} V_{\text{cd}} + V^*_{\text{tb}} V_{\text{td}} = 0$$

and normalizing it to $V_{\mathrm{cb}}^* V_{\mathrm{cd}}$:

$$1+\frac{V_{\mathrm{ub}}^* V_{\mathrm{ud}}}{V_{\mathrm{cb}}^* V_{\mathrm{cd}}}+\frac{V_{\mathrm{tb}}^* V_{\mathrm{td}}}{V_{\mathrm{cb}}^* V_{\mathrm{cd}}}=0$$

and using (2.50) and (2.51) we gain a triangle in the complex $\bar{\rho}$-$\bar{\eta}$ plane of the Wolfenstein parameters:

$$1-(\bar{\rho}+i\bar{\eta})-(1-\bar{\rho}-i\bar{\eta})=0.$$

which has comparable side lengths and no extreme acute angles:

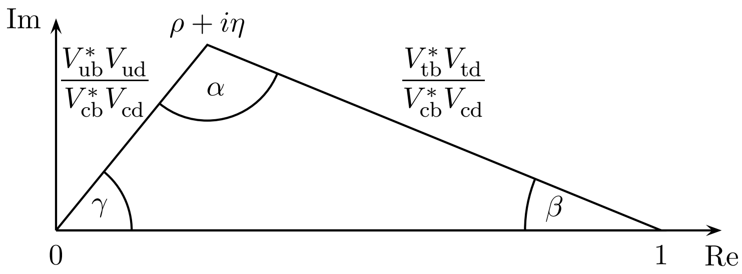


Figure 2.2: *Unitarity Triangle*

CKM-Fit

Figure 2.3 shows a fit of the CKM matrix by combining independent measurements of quantities connected to the matrix. These are the absolute values of the matrix elements, but also direct constraints on the angles α, β and γ and parameters coding the CP violation ϵ_K as well as mass differences Δm_s and Δm_d. All these measurements constrain the $(\bar{\rho},\bar{\eta})$ apex of the CKM triangle, indicated by the small blob shaded and bordered in red. It shows that so far all experiments are consistent with the CKM mechanism depicted by the CKM matrix fit. The experiments try to overconstrain the CKM matrix to test its unitarity, which needs high precisions in all determinations.

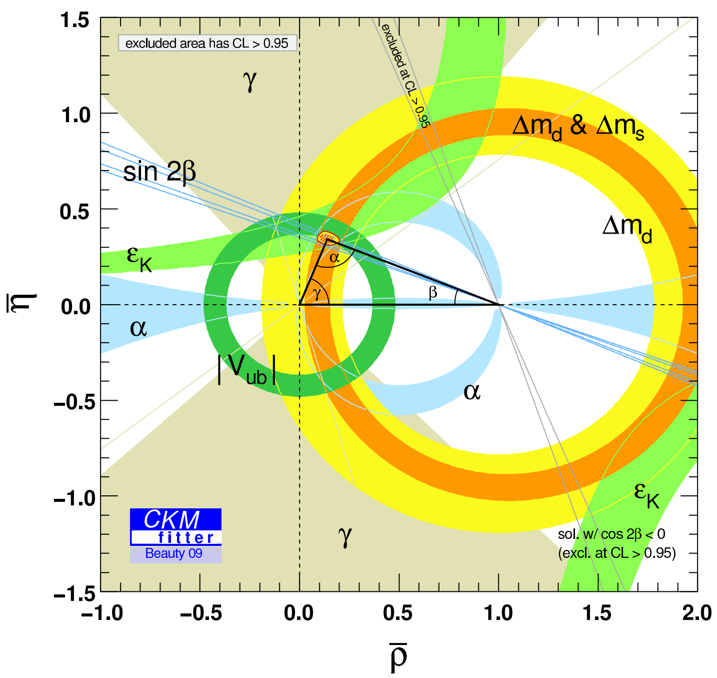


Figure 2.3: *Fit of the unitarity triangle*

2.3 Quantum Chromodynamics

Quantum chromodynamics (QCD) (see e. g. [4, 5]) is the part of the standard model describing the strong force, the interaction of quarks and gluons building a diversity of hadrons as QCD bound states. It is a non-abelian gauge theory, also called Yang-Mills theory, with the gauge group SU(3), in this context denoted as $SU(3)_C$, with "C" for color. Only the quarks interact strongly via the exchange of gluons, which are the gauge bosons of QCD. The gluons do not couple to the Higgs field and thus stay massless. The charge of the strong interaction is the so-called *color* and comes in three values *red*, *green* and *blue* (RGB). Thus a quark type exists in triplication with each realization of the color as a further quantum number. Therefore we have to add a color triplet to the quark wave function $\psi^q(x)$ to account for this extra symmetry:

$$\Psi^{1,q} = \psi^q(x) \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} , \quad \Psi^{2,q} = \psi^q(x) \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} \quad \text{and} \quad \Psi^{3,q} = \psi^q(x) \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \tag{2.53}$$

with the quark flavour index $q = u, d, c, s, t, b$ individually. Antiquarks have so-called anticolors. All hadrons, i. e. particles built up from quarks, are color neutral, which means that their wave function has a linear combination of color states with either the same amplitudes of all three colors (baryons) or in the form of a combination of color and anticolor (mesons). In further analogy to optics a baryon is said to be *white* with respect to the colors. In terms of group theory, the hadrons are color singlets. The choice of the *special* unitary transformations (SU(3) instead of U(3)) has the important consequence that there is no color singlet gluon, which would couple to hadrons. The result is that gluons stay within one hadron and have no infinite range, even though they are massless. Please note that in contrast the electromagnetism has infinite range due to the massless photon and weak interaction is short ranged because of the heavy masses of the gauge bosons $W^\pm$ and Z^0.

The $SU(3)_C$ is an exact local symmetry of the Lagrangian yielding no mass difference of quarks of different colors. The $SU(3)_C$ has 8 generators $T^a = \lambda^a/2$ with the Gell-Mann matrices λ^a. A gauge transformation is of the form

$$\Psi'^{\alpha,q} = \mathrm{e}^{-\mathrm{i}g_s\theta^a(x)T^a}\, \Psi^{\alpha,q} \tag{2.54}$$

where g_s is the strong coupling constant. The gauge bosons are 8 so-called *gluons* A^a_μ ($a = 1, \ldots ,8$) and the according covariant derivative reads

$$\mathrm{D}_\mu = \partial_\mu + \mathrm{i}g_s A^a_\mu T^a \tag{2.55}$$

The gluon fields do not commute leading to the name *non-abelian.* The field strength tensor of the gluon fields is

$$G^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g_s f^{abc} A^b_\mu A^c_\nu \tag{2.56}$$

with the structure constants f^{abc} of the SU(3):

$$[T^a,T^b] = \mathrm{i}f^{abc}T^c.$$

Contracting the gluon fields and the field strength tensor with the group generators ($A_\mu = A^a_\mu T^a$ and $G_{\mu\nu} = G^a_{\mu\nu}T^a$) allows us to write the contracted field strength as commutator of covariant derivatives:

$$G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - \mathrm{i}g_s\,[A_\mu, A_\nu] = \frac{i}{g_s}\,[\mathrm{D}_\mu\,, \mathrm{D}_\nu]\,. \tag{2.57}$$

The QCD Lagrangian with a summation over all quark flavours $q = u, d, c, s, t, b$ implied reads

$$\mathscr{L}_{\mathrm{QCD}} = \overline{\Psi}^{\alpha,q}\big(\mathrm{i}\not{D} - m_q\big)\Psi^{\alpha,q} - \frac{1}{4}G^a_{\mu\nu}G^{a\,\mu\nu}. \tag{2.58}$$

The kinetic term for the gluon fields can be rewritten in terms of $G_{\mu\nu} = G^a_{\mu\nu}T^a$ using $\mathrm{Tr}[T^aT^b] = 2\delta^{ab}$:

$$\mathrm{Tr}\,G_{\mu\nu}G^{\mu\nu} = G^a_{\mu\nu}G^{b\,\mu\nu}\,\mathrm{Tr}[T^aT^b] = 2G^a_{\mu\nu}G^{a\,\mu\nu}$$

yielding

$$\mathscr{L}_{\mathrm{QCD}} = \overline{\Psi}^{\alpha,q}\big(\mathrm{i}\not{D} - m_q\big)\Psi^{\alpha,q} - \frac{1}{2}\,\mathrm{Tr}\,[G_{\mu\nu}G^{\mu\nu}]\,. \tag{2.59}$$

The non-abelian structure in the kinetic term of the Lagrangian $-\frac{1}{2}\,\mathrm{Tr}\,[G_{\mu\nu}G^{\mu\nu}]$ amounts for a self-coupling of the gluons. They can self-interact in a three or four gluon vertex. This shows that they carry a color charge, which can be a combination of color and anticolor, because they exchange color charges between gluons. Also the gauge bosons of the electroweak theory self-interact, because they carry a weak hypercharge or in other words: the $\mathrm{SU}(2)_W$ is a non-abelian Lie group. The corresponding field strength tensor has a non-abelian structure as well.

2.3.1 Running Coupling, Asymptotic Freedom and Confinement

Calculations in quantum electrodynamics (QED) heavily rely on the smallness of the coupling of the electromagnetic interaction, which is described by the so-called fine-structure constant $\alpha = \frac{\mathrm{e}^2}{4\pi} \approx \frac{1}{137}$. It allows for the application of the perturbative expansion in α, enabling an order by order calculation of the effects from the interaction. This is normally done by the evaluation of Feynman diagrams via the application of Feynman rules of the theory. The leading order is called *tree-level*, because the Feynman diagrams have a “simple” structure similar to a tree. The next-to-leading order consist of one loop in the diagram, meaning a parallel combination of propagators with a new internal momentum running inside the loop. The evaluation of such a diagram requires to perform an integral of this additional internal momentum, often leading to divergences for either large momenta in the limit of infinity (ultra violet divergence) or decreasing small momenta (infrared divergence) with the limit zero. Higher orders are categorized by their loop count. These divergences nearly sentenced the quantum field theory approach to death, but was solved by the so-called renormalization, which we will also explain in the following. Contributions from higher orders are very small due to $\alpha \ll 1$ and the tree level result normally gives a very good description of the process in consideration.

Asymptotic Freedom In QCD this picture changes considerably, because its coupling α_s defined by the QCD gauge coupling constant g_s (the pendant to the electric charge e of QED)

$$\alpha_s = \frac{g_s^2}{4\pi}$$

is not necessary $\ll 1$. An important discovery is that α_s depends on the momentum transfer Q^2: $\alpha_s(Q^2)$. In QCD α_s decreases with growing Q^2 and in the limit $Q^2 \to \infty$ the coupling vanishes,

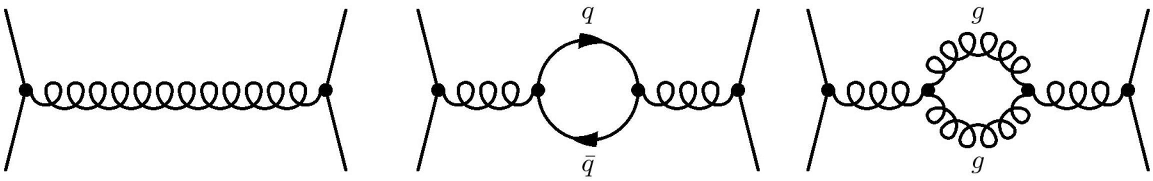


Figure 2.4: *Scattering of two quarks in lowest order and with quark-antiquark and gluon loop corrections*

which is known as *asymptotic freedom*. This is also the case for the QED fine structure constant α, but its dependency is very weak and in opposite direction: α grows with Q^2. The reason for the running coupling in QED is the *screening effect* from vacuum polarization. E. g. an electron carries a cloud with virtual photons and virtual electron-positron pairs. These pairs polarize the vacuum around the electron like a charge in a dielectric, screening the charge to a probe. With higher Q^2 a probe can better penetrate the cloud coming closer to the charge experiencing a stronger influence. The same screening effect of color charge with quark-antiquark pairs around a quark appears in QCD, too. However, the cloud of virtual gluons carry charges as well, due to the non-abelian structure of the gluon fields, which augments the color charge of the real quark. This is sometimes called *antiscreening effect* and is competitive to the screening effect. The effect on α_s can be calculated by the scattering of two quarks with the corrections of a quark-antiquark loop ($q\bar{q}$) and a gluon loop (gg) in the gluon exchange propagator (see fig. 2.4) with the separate results:

$$\alpha_s^{q\bar{q}}(Q^2) = \alpha_s(Q_0)\left(1 + 2N_f\frac{\alpha_s(Q_0^2)}{12\pi}\log\frac{Q^2}{Q_0^2}\right) \tag{2.60}$$

$$\alpha_s^{gg}(Q^2) = \alpha_s(Q_0)\left(1 + 11N_c\frac{\alpha_s(Q_0^2)}{12\pi}\log\frac{Q^2}{Q_0^2}\right) \tag{2.61}$$

with N_f the number of quark flavours, N_c the number of colors and Q_0^2 is the reference scale, where the tree-level process is fixed to having no corrections, because $\alpha_s(Q_0^2)$ is assumed to include them at this scale (see renormalization below). Combining both result and summing up all orders of the calculation, i. e. multiple insertions of quark-antiquark pairs and gluon pairs leading to a geometric series, yields

$$\alpha_s(Q^2) = \frac{\alpha_s(Q_0^2)}{1 + (11N_c - 2N_f)\frac{\alpha_s(Q_0^2)}{12\pi}\ln\frac{Q^2}{Q_0^2}}. \tag{2.62}$$

It is obvious that the change of $\alpha_s(Q^2)$ with Q^2 depends on the sign of $(11N_c - 2N_f)$. In QCD the number of colors is three and thus α_s decreases with raising Q^2 as long as there are less than 17 quark flavours. In the standard model there are 6 quark flavours leading to the mentioned *asymptotic freedom*.

Dimensional Transmutation, Confinement The problem lies on the opposite side, for small Q^2, where α_s increases considerably and even goes to infinity with

$$Q^2 \to \Lambda_{\mathrm{QCD}}^2 \qquad \text{with} \qquad \Lambda_{\mathrm{QCD}}^2 = Q_0^2 \exp\left(\frac{-12\pi}{(11N_c - 2N_f)\alpha_s(Q_0^2)}\right) \tag{2.63}$$

The parameter Λ_{QCD} is introduced to obtain a subtraction-point independent constant with the dimension of mass. Its value can only be determined from experiment and is roughly

$$\Lambda_{\mathrm{QCD}} \approx 200 - 300\,\mathrm{MeV}.$$

With this new "constant" we can rewrite the running of α_s as

$$\alpha_s(Q^2) = \frac{12\pi}{(11N_c - 2N_f)\ln\frac{Q^2}{\Lambda^2_{\mathrm{QCD}}}}. \tag{2.64}$$

At the scale of Λ_{QCD} we can consider the QCD to become strongly coupled, where perturbation theory breaks down and non-perturbative effects become important. It is interesting to note that this intrinsic scale Λ_{QCD} arises even in the case of massless quarks. The occurrence of a dimensionful parameters in a theory of only dimensionless ones is a specific property of QCD and known as *dimensional transmutation*.

Quarks and gluons in the final state of a process will always form hadrons which are color singlets, which is called hadronization. The postulate of the QCD that all observable states have to be color-neutral is called *color confinement*. The confinement has not yet been proven strictly, because of limited knowledge of QCD in the non-perturbative regime, but all experiment and lattice QCD calculations support confinement.

2.3.2 Renormalization

We mentioned that already in higher order calculations of processes divergences arise, due to quantum loop corrections. To deal with these divergences, they need to be *regularized*, which means to manifest the singularities in parameters and then to redefine fields and parameters of the theory to include the poles in their definition, which is known as *renormalization* (e. g. [6])

Regularization The basic property of regularization is to express the singularity of divergences in terms of parameters that are unphysical and should drop out in expressions of physical observables like cross sections. A variety of regularization methods have been invented, such as: cut-off regularization, where the integration is limited to a cut-off parameter; Pauli-Villars regularization, introducing fictitious massive particles as regulator like a mass for the photon. We will use this in the calculation of the radiative corrections to regulate the infrared divergences; dimensional regularization, continuating and solving the integral in a non-integer space-time dimension $D = 4 - 2\epsilon$ differing by 2ϵ from 4. The divergence will then appear as poles in ϵ like $1/\epsilon$ in the limit $\epsilon \to 0$. Either these poles will drop out in the physical observable or be removed by renormalization. Cut-Off regularization is not gauge invariant, while dimensional regularization is. Dimensional regularization has to deal with problems in defining the Dirac matrix γ^5 in D dimensions. Lattice QCD with a complete different approach finally provides another regularization through the grid size, which also limits the possible momenta.

Renormalization If regulators do not drop out in physical observables we have to reinterpret the observable in a way to include the divergence in the definition of an observable. This means for example that the emission and reabsorbation of photons of an electron can be seen as a being part of the electron and with that a contribution to its mass. Similar procedures are done for the electric charge or coupling constants in general and particle fields known as renormalization.

The quantities without radiative corrections included are called *bare quantities* and the idea is that they are not physical, the so-called *renormalized* quantities including the divergences from radiative corrections are the true observable quantities.

The renormalization describes the redefinition of the quantities. It is common to use the multiplicative renormalization, where the bare and renormalized quantities differ by a multiplicative renormalization constant Z, here for the quantities of the QCD part of the Lagrangian (2.59) with suppressed color indices α:

$$\Psi_0^q = \sqrt{Z_q}\,\Psi^q \tag{2.65}$$

$$m_{0q} = Z_m m_q \tag{2.66}$$

$$A_{0\mu}^a = \sqrt{Z_A} A_\mu^a \tag{2.67}$$

$$g_{0s} = Z_g g_s \mu^\epsilon \tag{2.68}$$

where the bare quantities are indicated by a 0 in the index. The term μ^ϵ in the renormalization definition of the strong coupling constant g_s was introduced to make it a dimensionless quantity in $D = 4 - 2\epsilon$. A very convenient way to deal with renormalization is the introduction of counterterms in the Lagrangian. The bare quantities in the Lagrangian are substituted by the renormalized ones and additionally the telescope trick is used to obtain separate terms with and without renormalization constants. E. g. for the Dirac term of the quark fields:

$$\begin{aligned}\mathscr{L}_\Psi =& \overline{\Psi}_0^q(\mathrm{i}\not{\partial} - m_{0q})\Psi_0^q = Z_q\overline{\Psi}^q\mathrm{i}\not{\partial}\Psi^q - Z_q Z_m m_q\overline{\Psi}^q\Psi^q \\ =& \overline{\Psi}^q(\mathrm{i}\not{\partial} - m_q)\Psi^q + (Z_q - 1)\overline{\Psi}^q\mathrm{i}\not{\partial}\Psi^q - (Z_q Z_m - 1)m_q\overline{\Psi}^q\Psi^q\end{aligned} \tag{2.69}$$

and expressing the renormalization constants as deviations from unity ($Z{=}1{+}\delta Z$, $Z_m m_q{=}m_q{+}\delta m_q$):

$$\mathscr{L}_\Psi = \overline{\Psi}^q(\mathrm{i}\not{\partial} - m_q)\Psi^q + \delta Z_q\overline{\Psi}^q(\mathrm{i}\not{\partial} - m_q)\Psi^q - \delta m_q\overline{\Psi}^q\Psi^q. \tag{2.70}$$

Thus the Lagrangian has a part similar to the original Lagrangian but only using renormalized quantities and a second part, the counter terms, involving the renormalization constants, which can be treated as additional interaction term. The renormalization constants are determined to cancel the divergences in the calculation of the radiative corrections, which depends on the renormalization scheme.

The application of dimensional regularization requires the introduction of a scale parameter μ to ensure the mass dimension of the integral, when altering the dimension of the momentum integration. The result depends on this scale parameter μ, which is the so-called renormalization scale, logarithmically as of $\log\frac{\mu^2}{Q^2}$, where Q^2 is a momentum transfer in the process. The renormalized quantities are fixed at the renormalization scale to have no contributions from loop corrections. Thus μ is chosen to be at the typical momentum transfers of the process to keep the logarithms and with that the loop contribution small. The renormalization scale is an unphysical parameter, which should drop out when summing up all orders of pertubation theory.

Renormalization Schemes

The redefinition of bare quantities to physical quantities is not unique. As the freedom of the scale μ to fix the renormalized quantities there is a freedom to choose the renormalization condition. From the point of view of the loop calculation the MS bar scheme is simple to apply, because as mentioned above the divergences of the integrals appear as $1/\epsilon$ poles after dimensional regularization, and in the *minimal subtraction* (MS) scheme these pole terms are just removed (subtracted) and assumed to be absorbed into the corresponding quantity, which is equal to set $Z_g \propto \frac{1}{\epsilon}$. These poles are accompanied by other constants:

$$\Delta_{\text{div}} = -\frac{1}{\epsilon} + 4\pi - \gamma_{\text{E}}$$

where $\gamma_{\text{E}} \approx 0.57721$ is the Euler–Mascheroni constant. Subtracting Δ_{div} instead of simply $1/\epsilon$ is known as the *modified minimal subtraction* or $\overline{\text{MS}}$.

In the on-shell scheme the renormalization condition are that the renormalized quantities have to be the physical ones, like the mass or the electric charge. Residuums of two-point functions have to be 1.

Renormalization Group Equation

A central role in perturbative QCD plays the *renormalization group equation.* which controls the dependence of the renormalization parameters on the renormalization scale. The derivation can be done by the requirement that the bare quantities must not depend on the renormalization scale. For the strong coupling constant this yields:

$$0 = \frac{\mathrm{d}g_{0s}}{\mathrm{d}\mu} = \frac{1}{\mu}\frac{\mathrm{d}g_{0s}}{\mathrm{d}\log\mu} \quad \Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\log\mu} Z_g g_s(\mu)\mu^{\epsilon} \tag{2.71}$$

$$\Rightarrow \quad \frac{\mathrm{d}}{\mathrm{d}\log\mu} g_s(\mu) = -\epsilon g_s \underbrace{-g_s \frac{1}{Z_g}\frac{\mathrm{d}Z_g}{\mathrm{d}\log\mu}}_{\beta(g_s)} = -\epsilon g_s + \beta(g_s), \tag{2.72}$$

where we defined the β function, which can be calculated by evaluating the $1/\epsilon$ part of the renormalization constant Z_g obtaining at one loop level in the limit of four dimensions:

$$\beta(g_s) = -\beta_0 \frac{g_s^3}{16\pi^2}. \tag{2.73}$$

with $\beta_0 = \frac{11N_c - 2N_f}{3}$. Rewriting the expression in terms of $\alpha_s = \frac{g_s}{4\pi}$ yields

$$\frac{\mathrm{d}\alpha_s}{\mathrm{d}\log\mu} = -2\beta_0 \frac{\alpha_s^2}{4\pi} \tag{2.74}$$

The differential equation (2.74) can be solved by separation of variables:

$$\int_{\infty}^{\alpha_s(\mu)} \frac{\mathrm{d}\alpha_s'}{{\alpha_s'}^2} = -\frac{\beta_0}{2\pi}\int_{\log\Lambda_{\text{QCD}}}^{\log\mu} \mathrm{d}(\log\mu)' \tag{2.75}$$

assuming $\alpha_s(\Lambda_{\text{QCD}}) \to \infty$, as discussed above. The result for the *renormalization group equation*

$$\alpha_s(\mu) = \frac{4\pi}{\beta_0 \log\frac{\mu^2}{\Lambda_{\text{QCD}}^2}} = \frac{12\pi}{(11N_c - 2N_f)\frac{\mu^2}{\Lambda_{\text{QCD}}^2}} \tag{2.76}$$

is again the description of the running coupling constant as already obtained in (2.64) with $\mu^2 = Q^2$. The calculation of β_0 consists exactly of the contribution of quark-antiquark and gluon loops to the quark-quark scattering as discussed in section 2.3.1.

Mass and Wave Function Renormalization

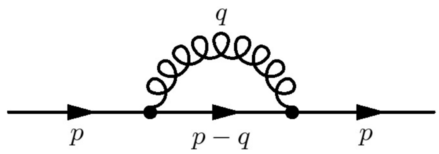


Figure 2.5: *Quark Self energy on one loop level*

For the renormalization of the masses and wave functions we have to compute the quark self-energy as $\Sigma(p^2)$ and consider the quark propagator from the Lagrangian with counter terms (2.71). Defining a physical mass for the quark is a problem, because quarks are confined in hadrons and can thus not been measured independently. We will start from the pole mass definition and move to the so-called *kinetic mass* for heavy quarks, which is more convenient in the HQE of semileptonic B decays. The renormalization condition for the pole mass is that the pole of the quark propagator should be at $p^2 = m_q^2$, being the "physical mass" of the quark:

$$\left(\mathrm{i}(\not{p} - m_q) + \mathrm{i}\Sigma(m^2) + \mathrm{i}Z_q(\not{p} - m_q) - \mathrm{i}\delta m_q\right) q(p) \stackrel{!}{=} 0 \tag{2.77}$$

The quark self-energy can be decomposed into a vector part Σ_V and a scalar part Σ_S:

$$\Sigma(p^2) = \not{p}\Sigma_V(p^2) + m_q\Sigma_S(q^2) \tag{2.78}$$

The application of the Dirac equation $(\not{p} - m_q)u(p) = 0$ disposes the $\mathrm{i}(\not{p} - m_q)$ terms in (2.77) and turns the $\not{p}$ in front of Σ_V into a m_q. The result defines the mass counter term in the pole scheme:

$$\frac{\delta m_q}{m_q} = \Sigma_V(m^2) + \Sigma_S(m^2) \tag{2.79}$$

It has been shown [7] that there is an ambiguity of about $\Lambda_{\mathrm{QCD}}/m_{\mathrm{b}}$ in the definition of the bottom quark pole mass, because it has to be related to physical quantities like the B meson mass. The result is a poor behavior of the perturbation series using the pole mass, even at second order. The kinematic mass scheme [8] resolves this problem by introducing a factorization scale μ_f and removing contributions from below this scale from the mass definition. The mass is then defined by a non-relativistic sum rule for the kinetic energy and at one-loop level the kinetic mass is related to the pole mass by

$$m_{\mathrm{q}}^{\mathrm{kin}}(\mu_f) = m_{\mathrm{q}}^{\mathrm{pole}}\left[1 - \frac{4}{3}\frac{\alpha_s}{\pi}\left(\frac{4}{3}\frac{\mu_f}{m_b} + \frac{\mu_f^2}{2m_{\mathrm{b}}^2}\right)\right] \tag{2.80}$$

The factorization scale is set to 1 GeV since this is the typical energy release in the process. Using this mass will reduce radiative corrections considerably leading to a better behavior of the perturbation series.

The renormalization condition for the quark wave function is that the residuum of the quark propagator has to be 1 at $\not{p} = m$, which can be displayed by

$$\left(\lim_{p^2 \to m^2} \frac{\mathrm{i}}{\not{p} - m} \Sigma(p^2) \right) u(p) \stackrel{!}{=} 0. \tag{2.81}$$

Using again the decomposition of the quark self energy (2.78) yields

$$0 = \delta Z_q + \Sigma_V(m^2) + \lim_{p^2 \to m^2} \frac{m(\not{p} + m)}{p^2 - m^2} \left(\Sigma_V(p^2) - \Sigma_V(m^2) + \Sigma_S(p^2) - \Sigma_S(m^2) \right).$$

The term with the limit excluding the nominator $m(\not{p} + m)$ is a differential quotient giving the derivative of $\Sigma_V(p^2) + \Sigma_S(p^2)$ with respect to p^2 at the point $p^2 = m^2$:

$$0 = \delta Z_q + \Sigma_V(m_q^2) + 2m_q^2 \frac{\mathrm{d}}{\mathrm{d}p^2} \left(\Sigma_V(p^2) + \Sigma_S(p^2) \right) \Big|_{p^2 = m^2} \tag{2.82}$$

$$\Rightarrow \quad \delta Z_q = - \Sigma_V(m_q^2) - 2m_q^2 \left(\Sigma'_V(m^2) + \Sigma'_S(m^2) \right) \tag{2.83}$$

2.4 Motivation

The B meson is a quark-antiquark bound state consisting of at least one bottom or anti-bottom quark. The second valence quark can be all other quarks except the top quark, because it decays before building a bound state. The mesons with the light up and down quarks are named $B^+ = u\bar{b}$, $B^- = \bar{u}b$, $B^0 = d\bar{b}$ and $\bar{B}^0 = \bar{d}b$. B mesons with strangeness and charmed B mesons are labeled by a corresponding subscript: $B_s^0 = s\bar{b}$, $\bar{B}_s^0 = \bar{s}b$, $B_c^+ = c\bar{b}$ and $\bar{B}_c^- = \bar{c}b$. The bound system of a bottom and anti-bottom quark is called the Upsilon meson $\Upsilon = b\bar{b}$. The $4s$ exited state of the Upsilon meson plays an important role at the so-called B factories Babar and Belle, because they decay mainly in a mixture of B^+B^- and $B^0\bar{B}^0$ pairs at the relative production rate $f_0 = 0.491 \pm 0.007$. This mixture is called $B\bar{B}$ and the difference to equal production stems from isospin violation. In this work we deal mostly with inclusive decays, where the important decay is the $b \to c$ transition. The light quark is of minor importance, which is also expressed in the naming *spectator quark*. For the analysis with experimental data from Babar, which we will use, we have to take into account and averaged lifetime of the charged and neutral B mesons ($\tau_\pm$ and τ_0) for the average state, which we will denote as B:

$$\tau_B = f_0\tau_0 + (1 - f_0)\tau_\pm = (1.585 \pm 0.007)\,\text{ps}. \tag{2.84}$$

The decay of a B meson is displayed in Fig. 2.6 on the left side, while the underlying quark decay is shown on the right side.

2.4.1 Complete Michel Parameter Analysis of Inclusive Semileptonic $b \to c$ Transition

Heavy flavour physics is nowadays in a mature state in both experimental and theoretical determinations. The B factories Babar and Belle produce an enormous amount of B mesons enabling precision determination of its decay properties. Especially the semileptonic decay $B \to X_c l\nu$ has been measured and studied extensively together with high developed theoretical tools as the heavy-quark expansion (HQE), which gives a very good description of the semileptonic decay mode. Also the radiative corrections have been calculated up to order α_s^2. This combination enabled a precision determination of the HQE parameter. Of particular interest is the CKM matrix element $|V_{cb}|$, with the most precise determination in fact coming from semileptonic B decays.

Besides the precision determination it is also possible to test for effects from physics beyond the standard model. Normally such effects are expected to show up in processes that have a tiny contribution in the standard model like flavour-changing neutral currents (FCNCs), suppressed

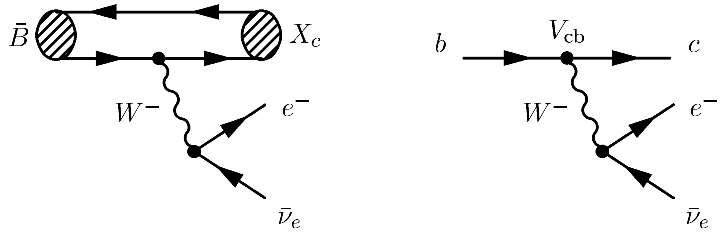


Figure 2.6: *B meson decay and underlying quark decay*

by the GIM mechanism. As FCNCs are only loop induced it is expected that they are sensitive to effects from virtual high-mass states.

Semileptonic B decays have a standard model contribution at tree-level, but the absence of right-handed currents in the standard model allows us to test for such a contribution from new physics. Such a test is known from the purely leptonic sector as *Michel parameter analysis* [9]. In 1957 Louis Michel calculated the differential decay rate of the process $\mu^+ \to e^+\nu_e\bar{\nu}_\mu$ in dependence of four parameters ρ, η, ξ and δ parametrizing the weak current structure. A left-handed current as proposed by Feynman and Gell-Mann with the V-A-theory [10] corresponds to $\rho = 3/2$, $\eta = 0$, $\xi = 1$ and $\delta = 3/4$. Of these so-called *Michel parameters* ρ is the most sensitive one to the current structure and a recent determination [11] gave $\rho = 0.751 \pm 0.001$ consistent with the prediction of 3/2.

We will perform an analysis analogously to the Michel parameter analysis for the $b \to c$ and the $e \to \nu$ current in semileptonic B decays. First we derive a model independent current from effective field theory considerations, sorting out non-standard model leptonic currents and introducing parameters similar to the Michel parameters for the $b \to c$ quark current. The enhanced current involves not only left- and right-handed vector currents, but also scalar and tensor couplings. As the state-of-the-art analysis of semileptonic B decays is a fit of the HQE parameters to the moments of the lepton energy spectrum and the moments of the hadronic invariant mass, we will compute these moments with the generic quark current in order to redo the fit including our new "Michel parameters".

Computing the moments up to order $1/m_\mathrm{b}^2$ and α_s it will turn out that the moments are not sensitive to the scalar and tensor currents, but only to the left- handed and right-handed vector currents. Thus we redo the combined fit with only the extension of a right-handed contribution to the $b \to c$ current. The fit will turn out to be consistent with a standard model left-handed current, but unfortunately the fit will reveal a low sensitivity to the right-handed currents as well. As exclusive decays are competitive in the determination of $|V_{cb}|$ we also determine a limit for possible right handed currents from them and analyse if a tension between the $|V_{cb}|$ values from inclusive and exclusive decays can thus be explained.

2.4.2 Computation of the α_s/m_b^2 Corrections to the Inclusive Decay $\overline{B} \to X_c\ell\overline{\nu}_\ell$

The calculation of the inclusive B decay involves two expansions: the perturbative expansion in α_s and the power expansions of the HQE in $1/m_\mathrm{b}$:

$$\text{Perturbative Corrections:} \quad \Gamma = \Gamma^{(0,0)} + \frac{\alpha_s}{\pi}\Gamma^{(1,0)} + \frac{\alpha_s^2}{\pi^2}\Gamma^{(2,0)} + \dots \tag{2.85}$$

$$\text{Heavy-Quark Expansion:} \quad \Gamma = \Gamma^{(0,0)} + \frac{1}{m_b^2}\Gamma^{(0,2)} + \frac{1}{m_b^3}\Gamma^{(0,3)} + \frac{1}{m_b^4}\Gamma^{(0,4)} + \dots \tag{2.86}$$

$$\text{Combined Expansion:} \quad \Gamma = \Gamma^{(0,0)} + \frac{\alpha_s}{\pi}\Gamma^{(1,0)} + \frac{1}{m_b^2}\Gamma^{(0,1)} + \frac{\alpha_s}{\pi m_b^2}\Gamma^{(1,2)} + \dots \tag{2.87}$$

The heavy-quark expansion has been developed a long time ago [12–15] and is well understood. The $\mathcal{O}(\alpha_s)$ have been computed only a few years ago [16–18]. The full $\mathcal{O}(\alpha_s^2)$ calculation has been performed recently [19]. The $\mathcal{O}(1/m_\mathrm{b}^4)$ were published in [20] and the full $\mathcal{O}(1/m_\mathrm{b}^5)$ will be published soon. Of the $\mathcal{O}(\alpha_s^3)$ only results for special kinematic points are calculated and the

BLM corrections. The orders of performed calculation can be visualized in the following matrix with the columns representing the order in $1/m_{\mathrm{b}}$ and the rows the order in α_s: Please note

		$1/m_b^n$				
	n	0	2	3	4	5
	0	$\bullet$	$\bullet$	$\bullet$	$\bullet$	$\circ^{a}$
α_s^n	1	$\bullet$	$\circ^{b}$	–	–	–
	2	$\bullet$	–	–	–	–
	3	$\circ^{c}$	–	–	–	–

Table 2.4: *Calculated orders in α_s (rows) and HQE (columns) for the inclusive decay $\bar{B} \to X_c \ell \bar{\nu}_\ell$. ([a] Will be published soon. [b] Only the correction to the kinetic energy operator μ_π^2. [c] Only at special kinematic point and the $\alpha_s^3 \beta_0^2$-BLM corrections.)*

that the HQE starts at $\mathcal{O}(1/m_{\mathrm{b}}^2)$ and thus the expansion parameters are roughly of the same size ($\frac{\alpha_s}{\pi} \approx \frac{1}{m_{\mathrm{b}}^2}$) indicating that the contributions of the same order of magnitude form more or less a triangle in the upper left corner of the matrix. This suggests that the combined $\alpha_s/m_{\mathrm{b}}^2$ corrections are crucial for further increasing the precision in the theoretical description of the semileptonic $\bar{B} \to X_c \ell \bar{\nu}_\ell$ decay, prior to the computation of further orders in α_s or $1/m_{\mathrm{b}}$. At the order $1/m_{\mathrm{b}}^2$ exist two HQE parameter: the kinetic energy operator $\hat{\mu}_\pi^2$ and the chromomagnetic operator $\hat{\mu}_{\mathrm{G}}^2$. The α_s corrections of these operators correspond to the full $\alpha_s/m_{\mathrm{b}}^2$ calculation. The α_s corrections to $\hat{\mu}_\pi^2$ have been evaluated recently in [21]. It is possible to determine these corrections easily via reparametrization invariance, which is also mentioned in [21]. We will present this determination in chapter 7.1 because it was obtained at the same time as [21] and by communicating with the authors. Nevertheless, the radiative corrections to $\hat{\mu}_{\mathrm{G}}^2$ have not yet been computed. The know-how in both the heavy quark expansion and the perturbative expansion in the semileptonic B meson decay tempts us to go about this calculation. In anticipation of this work we present in 7.2 the calculation of certain real corrections of $\hat{\mu}_\pi^2$ to the hadronic invariant mass, which are possible without virtual corrections. We expect the method presented therein, to be a good candidate for the calculation of the corrections to $\hat{\mu}_{\mathrm{G}}^2$.

2.4.3 Structure of the document

After the introduction to the standard model of particle physics and this motivation we will introduce the basics of effective theories in section 3.1, which we need in the introduction to the heavy-quark effective theory (3.2) and the derivation of the generic weak currents (3.3), which are all part of the chapter 3 about effective theories. The following chapter 4 introduces the heavy-quark expansion (HQE), which is closely related to the HQET, and applies it using the generic current derived in 3.3 to the inclusive decay $\bar{B} \to X_c \ell \bar{\nu}_\ell$ computing the moment of the lepton energy spectrum and the moment of the hadronic mass depending on the coupling. In chapter 5 we extend the same calculation of moments to include the $\mathcal{O}(\alpha_s)$ corrections, including a discussion and computation of the operator mixing which occurs. Finally in the weak current analysis, we perform the combined HQE fit including a possible right-handed contribution in chapter 6. Chapter 7 presents the calculation of the radiative corrections to $\hat{\mu}_\pi^2$ via reparametrization invariance (7.1) and in a full calculation certain moments of the hadronic invariant mass, which are possible to calculate without virtual corrections as a road to the evaluation of the α_s corrections to $\hat{\mu}_{\mathrm{G}}^2$. In the end in chapter 8 we summarize our results and conclude.

3 Effective Theories

3.1 Properties of Effective Theories

For the description and calculation of a physical system the focus lies on the relevant degrees of freedom appropriate for the problem at hand. The relevant degrees of freedom are defined by the distance or energy scale of the problem at hand. Even though quantum mechanics or special relativity are a more fundamental descriptions of nature than classical mechanics, computing all physical problems with these *full theories* would increase the complexity of the problem with almost no precision yield. Classical mechanics is an effective theory of quantum mechanics in the limit of long distances and of special relativity for low velocities.

Characteristic of effective theories is that degrees of freedom not relevant for the typical energy or distance scale of the problem do not appear explicitly in the Lagrangian. Typically it is good to construct an effective theory in presence of very disparate scales. In particle physics a typical use of an effective theory is the calculations of interactions involving heavy particles at low energy scales. With typical energy scales below the rest mass of the heavy particle it cannot appear in the final state and thus not as a degree of freedom for the effective Lagrangian at this scale. Such a heavy particle can appear as a virtual, intermediate particle and thus contributes to the process. Its effect appears suppressed by inverse powers of the heavy mass. These properties are described by the decoupling theorem by Applequist and Carazzone [22].

Examples for the use of effective theories are

- *Fermi Theory:* Weak decays are mediated in the standard model by the heavy $W^{\pm}$ and Z bosons. E. g. for the decay of the bottom quark the relevant energy scale is much lower than the mass of the W boson which can thus only appear as a virtual particle. In the Fermi theory the degrees of freedom of the W boson are removed from the Lagrangian yielding a four-fermion interaction with the effect of the heavy boson suppressed by the inverse power of its mass. We will go in more details in section 3.1.2.
- *Heavy-Quark Effective Theory:* The interaction of the B meson is described in the heavy-quark effective theory as a free quark with corrections suppressed by inverse powers of the bottom quark mass. These corrections are the interaction with the second light quark in the meson.
- *Standard Model:* Considering New Physics the standard model is assumed to be the effective theory at the energy range currently accessible by the particle accelerators. The specific model describes the interactions, particles and the typical scale of this assumed *full theory.* It is believed that the scale is fixed upwards by the Planck scale.

The advantages of using effective theories are the simplification of the calculation and the summation of large terms involving logarithms of the disparate scales that appear in the full theory. The summation can be performed by the renormalization group equation of the effective theory.

3.1.1 Separation by a large scale

The *full theory* describes the transition from an initial state $|i\rangle$ to a specific final state $|f\rangle$ by their matrix element with the Hamiltonian of the full theory $\mathscr{H}_{\text{full}}$:

$$\langle f|\mathscr{H}_{\text{full}}|i\rangle.$$

To construct the effective theory a large scale Λ compared to the energies of the considered process $E_{\text{i,f}}$ has to be identified, typically the mass of a heavy particle in the full theory. The Hamiltonian of the effective theory has to have the same transition matrix element as the one of the full theory. The Hamiltonian can be written as a sum of local operators at the scale Λ:

$$\langle f|\mathscr{H}_{\text{eff}}|i\rangle = \sum_k C_k(\Lambda)\langle f|\mathcal{O}_k|i\rangle\bigg|_{\Lambda}. \tag{3.1}$$

The main feature of this expansion is, that short-distance physics below Λ are enclosed in the coefficients $C_k(\Lambda)$ and long-distance physics above Λ in the operators $\mathcal{O}_k$. Short-distance physics comprises effects of non-relevant degrees of freedoms as discussed above. These may be heavy particles which can only be produced intermediately or even new-physics effects above the scale Λ, parametrized depending on the model under consideration or in a general way, which we are going to pursue in this work for semileptonic B-decays. Long-distance physics describe the relevant degrees of freedoms at the probing scale. In the case of the Fermi Theory this is an effective four-fermion operator, which does not exist in the standard model being the full theory. In the case of the construction of effective theories for new-physics effects these may be the standard model operators. The separation of effects from above or below the dividing scale Λ only simplifies theory and calculations, if the series with $k \to \infty$ can be truncated for a finite k.

As the Hamiltonian is an energy density (energy/length^3) it has a mass dimension of four in natural units (energy $\sim$ mass and 1/length $\sim$ mass). Thus the combined dimension of the coefficients $C_k(\Lambda)$ and the matrix elements $\langle f|\mathcal{O}_k|i\rangle$ needs to be four in every order:

$$\dim\big(C_k(\Lambda)\big) + \dim\big(\langle f|\mathcal{O}_k|i\rangle\big) = 4.$$

It is convenient to factor out an appropriate power of Λ to make the coefficients dimensionless. The coefficients can only have mass dimension originating from powers of Λ, because they do not depend on any long distance scale. The inverse powers of a large scale Λ provide a convenient expansion parameter causing a substantial decrease in every order of this power series. Thus we order the inverse powers of Λ according to the index k. The suppression by Λ gives rise to matrix elements of higher dimensions than four. Accounting for the possibility that more than one operator contributes in every order we insert a second sum with the index i:

$$\langle f|\mathscr{H}_{\text{eff}}|i\rangle = \sum_k \frac{1}{\Lambda_K}\sum_i C_{k,i}\langle f|\mathcal{O}_k|i\rangle\bigg|_{\Lambda}. \tag{3.2}$$

The truncation of the sum for a finite k with operators of mass dimension n correspond to a neglection of terms with a power suppression of $1/\Lambda^{n-4}$. The power series includes the dimension four term. Thus the effective theory is renormalizable, because the dimension four part is renormalizable and any finite number of insertions of operators of higher dimensions does not harm the renormalizability.

3.1.2 Example: Fermi Theory

The full standard model interaction Hamiltonian of the weak decay partonic decay $b \to cl\bar{\nu}$ is

$$\mathscr{H}_{\mathrm{Int}}(b \to cl\bar{\nu}) = -\left(\frac{g_2}{\sqrt{2}}\right)^2 V_{cb}\big(\bar{u}(p_c)\gamma^\mu P_L u(p_b)\big)\big(\bar{u}(p_e)\gamma^\nu P_L v(p_\nu)\big)\frac{1}{q^2 - M_W^2}\left[g_{\mu\nu} - \frac{q_\mu q_\nu}{M_W^2}\right]. \quad (3.3)$$

with $q = p_\mathrm{b} - p_\mathrm{c}$. Despite the name "heavy quark" the bottom quark is considerably lighter than the intermediate W^- boson. Also the maximal momentum transferred by the W boson propagator $q^2_{\mathrm{max}} = (m_\mathrm{b} - m_\mathrm{c})^2$ is small compared to the W boson mass. Thus propagator denominator $q^2 - M_W^2$ can be approximated by $-M_W^2$ and the $q_\mu q_\nu / M_W^2$ term neglected. The W boson propagator in the effective theory then reads

$$-\frac{g_{\mu\nu}}{M_W^2}.$$

The resulting effective Hamiltonian exhibits a four-fermion interaction:

$$\mathscr{H}_{\mathrm{Int}}(b \to cl\bar{\nu}) = \frac{4\mathrm{G_F} V_{cb}}{\sqrt{2}}\big(\bar{u}(p_c)\gamma_\nu P_L u(p_b)\big)\big(\bar{u}(p_e)\gamma^\nu P_L v(p_\nu)\big) \quad (3.4)$$

with the Fermi constant

$$\frac{\mathrm{G_F}}{\sqrt{2}} = \frac{g_2^2}{8M_W^2}. \quad (3.5)$$

Historically the four-fermion coupling was proposed by Enrico Fermi prior to the discovery of the W boson and the corresponding theory of weak interactions. Likewise a lot of physical theories are first the proposed full theory and later turn out to be an effective theory of a more comprehensive theory.

The lowest order effective transition matrix element for the decay $b \to cl\nu$ is

$$\langle cl\bar{\nu}|\mathscr{H}_{\mathrm{eff}}|b\rangle = \frac{4\mathrm{G_F} V_{cb}}{\sqrt{2}}\big(\bar{c}\gamma_\nu P_L b\big)\big(\bar{l}\gamma^\nu P_L \nu\big) = \frac{4\mathrm{G_F} V_{cb}}{\sqrt{2}}\langle cl\bar{\nu}|(\bar{c}\gamma_\nu P_L b)(\bar{l}\gamma^\nu P_L \nu)|b\rangle \quad (3.6)$$

with the shortcuts $\bar{c} = \bar{u}(p_c)$, $b = u(p_b)$, $\bar{l} = \bar{u}(p_l)$, $\nu = u(p_\nu)$ for the particles' spinors. The matrix element in (3.6) on the right has the mass dimension 6, because fields and spinors have mass dimension 3/2 of which we have 4. The coefficient in this lowest term is of order $1/\Lambda$, which is here the mass of the W boson M_W included in the Fermi constant $\mathrm{G_F}$ (3.5). Thus we end up with dimension $6 - 2 = 4$, the correct mass dimension of the Hamiltonian.

3.2 Heavy-Quark Effective Theory

The heavy-quark effective theory is an effective theory for interactions involving heavy quark fields. The Hamiltonian or likewise the Lagrangian of the full standard model interaction is expanded in inverse powers of the heavy quark mass m_Q. Because of our later application we consider the decay of a heavy meson. The basic statement of the heavy-quark effective theory is that the heavy quark within a meson is almost on-shell and thereby mainly directing the meson's interactions. The typical energies of a decay are small compared to the heavy quark's mass and thus the so-called heavy degrees of freedom above at the scale above the heavy quark mass are not relevant for the theoretical description and can be integrated out. The heavy degrees of freedom turn out to have the double mass of the heavy quark corresponding to a heavy quark antiquark pair which can only appear intermediately in the decay.

In the setup of the effective theory the heavy quark field Q is separated from all other quark and gluon fields ϕ_λ. Besides the heavy quark field these fields represent all remaining degrees of freedom and therefore are called "light degrees of freedom". In the following we will discuss the heavy-quark effective theory in the context of the path integral formulation. Using the notation for the heavy quark field Q and the light degrees of freedom ϕ_λ the generating functional of the QCD reads

$$Z\{\eta,\bar\eta,\lambda\} = \int \mathcal{D}[Q]\,\mathcal{D}[\bar Q]\,\mathcal{D}[\phi_\lambda]\, e^{i\{S_Q+S_\lambda+\int \mathrm{d}^4x(\bar\eta Q+\bar Q\eta+\phi_\lambda\lambda)\}}$$

where

$$S_Q = \int \mathrm{d}^4x\,\bar Q\big(i\not{D} - m_Q\big)Q, \tag{3.7}$$

is the action of the heavy-quark field, including the interaction with gluon fields via the covariant derivative $D_\mu = \partial_\mu + ig_sA_\mu$. The η, $\bar\eta$, and λ are source terms of the heavy-quark field, the corresponding anti-quark field and the light degrees of freedom. As discussed the heavy degrees of freedom are not relevant and can be removed from the theory. The existence of source terms for fields in the functional integral correspond to possible initial or final states with particles of these fields. The roadmap for the derivation of the heavy-quark theory is thus to separate the fields into heavy and light parts and "integrate out" the heavy degrees of freedom and thereby removing the source terms for the heavy-degrees of freedom. Finally the heavy-quark field can be expressed in a power series in the inverse of the heavy-quark mass yielding also a power series of the corresponding Lagrangian and Hamiltonian. This will end up in an effective theory form as of (3.2).

We relate a heavy quark to a heavy meson containing a light anti-quark. The free meson has its momentum p_{meson} on-shell, which allows us to define its velocity as

$$v = \frac{p_{\text{meson}}}{m_{\text{meson}}}, \qquad v^2 = 1, \qquad v_0 > 0.$$

The heavy quark bound inside the meson has a momentum we decompose according to

$$p_Q^\mu = m_Q v^\mu + k^\mu.$$

The term $m_Q v^\mu$ is the on-shell momentum of a free particle with mass m_{Q} and the velocity v of the meson. The residual momentum k^μ describes the binding of the heavy quark with the light quark, which is mediated by gluons and virtual particle-anti-particle pairs. The heavy quark is not on-shell due to the binding and thus its invariant mass is not m_{Q}. The momentum k^μ is essentially the so-called *off-shell part* of the heavy-quark's momentum and includes not only the binding effects, but also the motion of the heavy quark inside the meson, because we used v^μ, the mesons velocity and not the quarks velocity in the term $m_Q v^\mu$. The binding energy is of order Λ_{QCD}, which is small to the rest mass m_{Q} of the heavy-quark. Thus the heavy quark inside the meson is close to being on-shell and the residual momentum k^μ is small yielding a convenient quantity for an approximation or an expansion. The expansion in k^μ is used in the heavy-quark expansion (HQE). Here we will use it to remove the heavy degrees of freedom from the theory.

The heavy quark field can be decomposed into two spinor components, of which one describes a quark propagating with the velocity v and the other an anti-quark with $-v$ and thus the vector

v can be used to project out these two components:

$$P_v^{\pm} = \frac{1}{2}(1 \pm \not{v}).$$

Additionally, using the rest frame of the meson the velocity is $v^\mu = (1,0,0,0)$. The $U(1)$ gauge symmetry allows us to redefine the phases of the fields by $e^{im_Q vx}$, which is in the meson's restframe a large time dependence with the frequency $m_Q c$, the inverse Compton wavelength of the heavy quark.

$$\begin{aligned} h_v &= e^{im_Q vx} P_v^+ Q = e^{im_Q vx} \frac{1}{2}(1+\not{v})Q \\ H_v &= e^{im_Q vx} P_v^- Q = e^{im_Q vx} \frac{1}{2}(1-\not{v})Q \end{aligned}$$

For the particle field $1/2(1+\not{v})Q$ with the phase $e^{-ip_Q x}{=}e^{-im_Q vx}e^{-ikx}$ this means a split-off of the strong oscillating term $e^{-im_Q vx}$, leaving only a small time dependence via the residual momentum k^μ. The same transformation of the anti-quark field's phase $e^{ip_Q x}$ yields nearly a duplication of the phase frequency $e^{i(2m_Q v+k)x}$, assuming k^μ is small compared to $m_Q v^\mu$. The transformed fields h_v and H_v exhibit a strong scale separation in regarding their massiveness, which will become clear by application of the Dirac equations, these fields fulfill.

It is convenient to decompose the covariant derivative into a longitudinal $v^\mu(v{\cdot}D)$ and a transversal part $D_\perp^\mu$:

$$D^\mu = v^\mu(v\cdot D) + D_\perp^\mu \quad \text{with} \quad v_\mu D_\perp^\mu = 0 \quad \text{and} \quad \left\{\not{D}_\perp, \not{v}\right\} = 0.$$

Using the decomposition of the heavy quark field $Q = H_v + h_v$ the action in (3.7) becomes

$$\begin{aligned} S_Q &= \int \mathrm{d}^4x (\bar{H}_v + \bar{h}_v) e^{im_Q vx} \left(i\not{D} - m_Q\right) e^{-im_Q vx}(H_v + h_v) \\ &= \int \mathrm{d}^4x (\bar{H}_v + \bar{h}_v)\left(i\not{D} + m_Q\not{v} - m_Q\right)(H_v + h_v) \\ &= \int \mathrm{d}^4x \left[(\bar{H}_v + \bar{h}_v)\left(i\not{D} - 2m_Q\right) H_v + (\bar{H}_v + \bar{h}_v)\left(i\not{D}\right) h_v\right], \end{aligned}$$

because $\not{v}H_v{=}{-}H_v$ and $\not{v}h_v{=}h_v$. Using the decomposition of the covariant derivative we have

$$\begin{aligned} = \int \mathrm{d}^4x \bar{H}_v &\left(i\not{v}(v\cdot D) + i\not{D}_\perp - 2m_Q\right) H_v + \bar{h}_v \left(i\not{v}(v\cdot D) + i\not{D}_\perp - 2m_Q\right) H_v \\ &+ \bar{H}_v \left(i\not{v}(v\cdot D) + i\not{D}_\perp\right) h_v + \bar{h}_v \left(i\not{v}(v\cdot D) + i\not{D}_\perp\right) h_v. \end{aligned}$$

Due to $\left\{\not{D}_\perp, \not{v}\right\}{=}0$ and $P_v^- P_v^+{=}0$ it holds

$$\bar{H}_v \not{D}_\perp H_v = \bar{Q} e^{-im_Q vx} P_v^- P_v^+ \not{D}_\perp e^{im_Q vx} Q = 0$$

as well as $\bar{h}_v \not{D}_\perp h_v = 0$ and because of $\not{v}h_v{=}h_v$:

$$\bar{H}_v i\not{v}(v\cdot D)h_v = \bar{H}_v i(v\cdot D)h_v = \bar{Q}e^{-im_Q vx}P_v^- i(v\cdot D)P_v^+ e^{im_Q vx}Q = 0,$$

and analogously $\bar{h}_v i\not{v}(v \cdot D)H_v = 0$. And with $\bar{h}_v H_v = 0$ the action is

$$S_Q = \int \mathrm{d}^4x \left[\bar{h}_v i(v \cdot D)h_v - \bar{H}_v\{i(v \cdot D) + 2m_Q\}H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v\right].$$

We can now express the source terms by H_v and h_v or $\bar{H}_v$ and $\bar{h}_v$, respectively:

$$\begin{aligned} &\int \mathrm{d}^4x \left\{\bar{\eta}e^{-im_Q vx}(H_v + h_v) + \eta e^{im_Q vx}(\bar{H}_v + \bar{h}_v) + \phi_\lambda \lambda\right\} \\ =& \int \mathrm{d}^4x \left\{\bar{\rho}_v h_v + \bar{h}_v \rho_v + \bar{R}_v H_v + \bar{H}_v R_v + \phi_\lambda \lambda\right\} \end{aligned}$$

with

$$\begin{aligned} \bar{\rho}_v &= \bar{\eta}e^{-im_Q vx} & \rho_v &= \eta e^{im_Q vx} \\ \bar{R}_v &= \bar{\eta}e^{-im_Q vx} & R_v &= \eta e^{im_Q vx}, \end{aligned}$$

as source terms for the fields h_v and H_v. The excitations of H_v are suppressed by the double quark mass $2m_\mathrm{Q}$ and appear only intermediately. Thus an outer source term is not needed for this field and can be switched off, making the path integral of this field executable. We use a generalization of the Gaussian integral for complex quadratic matrix forms:

$$\int e^{-(\mathbf{z}^+ A\mathbf{z} + B\mathbf{z} + \mathbf{z}^+ C + D)} \mathrm{d}^n z^* \mathrm{d}^n z = \frac{(2\pi i)^{n/2}}{\det A} e^{BA^{-1}C - D}.$$

Identifying $A{=}iv{\cdot}D{+}2m_Q$, $B{=}{-}\bar{h}_v i\not{D}_\perp$, $C{=}{-}i\not{D}_\perp h_v$ and the rest with D we obtain

$$S_Q = \int \mathrm{d}^4x \left[\bar{h}_v i(v \cdot D)h_v - \bar{h}_v \not{D}_\perp \left(\frac{1}{iv \cdot D + 2m_Q - i\epsilon}\right) \not{D}_\perp h_v\right], \tag{3.8}$$

and the generating functional in the new fields is

$$Z\{\rho_v, \bar{\rho}_v, \lambda\} = \int \mathcal{D}[h_v]\mathcal{D}[\bar{h}_v]\mathcal{D}[\phi_\lambda] \det{}^{-1}(iv \cdot D + 2m_Q)\, e^{i\{S_Q + S_\lambda + \int \mathrm{d}^4x(\bar{\rho}_v h_v + \bar{h}_v \rho_v + \phi_\lambda \lambda)\}}$$

The determinant $\det(iv \cdot D + 2m_Q)$ turns out to be constant and can be pulled in front of the integral. A constant factor in the generating functional has no effect on the physics and can be set to one.

Integrating out the fields H_v and $\bar{H}_v$ corresponds to the substitution

$$H_v = \frac{1}{iv \cdot D + 2m_Q} i\not{D}_\perp h_v.$$

The field H_v can thus be expressed by h_v. This seems reasonable, because h_v and H_v are spin components of one field, namely Q, which are not independent under the presence of an interaction.

3.2.1 Effective Lagrangian and Propagator

The expression for the generating functional is still exact, because we merely redefined fields without any approximation. The exponent in the path integral turns out to be non-local. This inverse operator can be expanded in local operators. A transformation into momentum space converts the derivatives into momenta, which are integrated. The expansion only converges for momenta below $2m_Q$, which is the typical range where HQET is applied. The field $Q(x)$ becomes:

$$\begin{aligned} Q(x) = e^{-im_Q vx}[h_v + H_v] &= e^{-im_Q vx}\left[1 + \left(\frac{1}{2m_Q + iv\cdot D}\right) i\not{D}_\perp\right] h_v \\ &= e^{-im_Q vx}\left[1 + \frac{1}{2m_Q}\left(1 - \frac{iv\cdot D}{2m_Q} + \dots\right) i\not{D}_\perp\right] h_v \\ &= e^{-im_Q vx}\left[1 + \frac{1}{2m_Q} i\not{D}_\perp + \left(\frac{1}{2m_Q}\right)^2 (-iv\cdot D) i\not{D}_\perp + \dots\right] h_v. \end{aligned} \tag{3.9}$$

The effective Lagrangian can be obtained by substituting the expansion of Q into the Lagrangian of the full theory:

$$\begin{aligned} \mathscr{L} &= \bar{h}_v(iv\cdot D)h_v + \bar{h}_v i\not{D}_\perp \left(\frac{1}{2m_Q + iv\cdot D}\right) i\not{D}_\perp h_v \\ &= \bar{h}_v(iv\cdot D)h_v + \frac{1}{2m_Q}\bar{h}_v(i\not{D}_\perp)^2 i h_v + \left(\frac{1}{2m_Q}\right)^2 \bar{h}_v(i\not{D}_\perp)(-iv\cdot D)(i\not{D}_\perp)h_h + \dots \end{aligned}$$

This is in lowest order $\mathscr{L}=\bar{h}_v(ivD)h_v$. In momentum space the propagator of the h_v field is:

$$\frac{1}{v\cdot k + i\epsilon},$$

because it only depends on the residual momentum k^μ of the heavy quark in the meson.

3.3 Generic Parametrization of New Physics in Quark Mixing

In the search for new physics there is often a specific model developed which needs to predict observables to test it with experimental data. But with the tools of effective field theories it is also possible to parametrize new physics generically without assuming a certain model. The advantage is that we can cover a large range of models by just searching for any effect in the generic parametrization and we can also restrict the type of new physics depending on the outcome of the new physics parameter. The problem with a generic parametrization arise from the fact that typically the number of new parameters is to high in order to fix them by fits to experimental data. Therefore it is crucial to reduce the number of parameters by identifying possible dependencies between them or finding relations to drop parameters. Applying a generic ansatz to a certain decay, like the inclusive $b \to cl\nu$ decay here, reduces the number of parameters considerably, as we will see.

The standard model is the most general renormalizable theory with the observed $\mathrm{SU}(2)_\mathrm{C} \times \mathrm{SU}(2)_\mathrm{L} \times U(1)_\mathrm{Y}$ symmetry and the observed particle spectrum. In the sense of effective field theories we discussed in the previous chapter 3 we assume new physics to be the *full theory* and the standard model the effective theory below the scale of the weak boson masses. For the Lagrangian this reads

$$\mathscr{L} = \mathscr{L}_{4\mathrm{D}} + \frac{1}{\Lambda}\mathscr{L}_{5\mathrm{D}} + \frac{1}{\Lambda^2}\mathscr{L}_{6\mathrm{D}} + \dots \tag{3.10}$$

where the lowest order represents the standard-model Lagrangian $\mathscr{L}_{\mathrm{SM}} = \mathscr{L}_{4\mathrm{D}}$ and Λ is the new physics' scale. With the standard model being the lowest order and most general model with operators of dimension four any effect of new physics has to appear at the scale of the weak boson mass as a set of operator of mass dimension six or higher, because it turns out that there are no dimension five operator respecting the observed $\mathrm{SU}(2)_{\mathrm{L}} \times U(1)_{\mathrm{Y}}$ symmetry. We will group the operator according to their helicity structure.

In the derivation we follow [23]. The roadmap in constructing the relevant dimension six parameters is to write down all possible combinations of relevant standard model fields giving dimension six operators, since we expand the new physics in standard model fields. The possible combinations fall into two classes: two-quark-two-lepton operators or two-quark fields with gauge and Higgs boson fields. The fermion fields have mass dimension 3/2 and the gauge and Higgs bosons dimension 1. Here we stick to a single Higgs boson model as assumed in the standard model, which has not yet been proven. The extension to a type-II Higgs doublet as needed e. g. in supersymmetry is straightforward.

We start from a $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry for bookkeeping reasons. We will allow to explicitly break the $\mathrm{SU}(2)_R$ symmetry. We group quark and lepton fields according to their handedness and family

$$Q_L = \begin{pmatrix} u_L \\ d_L \end{pmatrix}, \quad \begin{pmatrix} c_L \\ s_L \end{pmatrix}, \quad \begin{pmatrix} t_L \\ b_L \end{pmatrix} \qquad \text{for the left handed quarks} \tag{3.11}$$

$$Q_R = \begin{pmatrix} u_R \\ d_R \end{pmatrix}, \quad \begin{pmatrix} c_R \\ s_R \end{pmatrix}, \quad \begin{pmatrix} t_R \\ b_R \end{pmatrix} \qquad \text{for the right handed quarks} \tag{3.12}$$

$$L_L = \begin{pmatrix} \nu_{e,L} \\ e_L \end{pmatrix}, \quad \begin{pmatrix} \nu_{\mu,L} \\ \mu_L \end{pmatrix}, \quad \begin{pmatrix} \nu_{\tau,L} \\ \tau_L \end{pmatrix} \qquad \text{for the left handed leptons} \tag{3.13}$$

$$L_R = \begin{pmatrix} \nu_{e,R} \\ e_R \end{pmatrix}, \quad \begin{pmatrix} \nu_{\mu,R} \\ \mu_R \end{pmatrix}, \quad \begin{pmatrix} \nu_{\tau,R} \\ \tau_R \end{pmatrix} \qquad \text{for the right handed leptons} \tag{3.14}$$

We will suppress an index indicating the family in the following to keep the notation simple. The displayed operators are sums over all possible family combinations. The left-handed Q_L and L_L transform as (2,1) under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and the right-handed Q_R and L_R as (1,2). $\mathrm{SU}(2)_R$ The Higgs fields transforms as (2,2) under $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ and can be written as a 2 matrix:

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} \phi_0 + i\chi_0 & \sqrt{2}\phi_+ \\ \sqrt{2}\phi_- & \phi_0 - i\chi_0 \end{pmatrix} \tag{3.15}$$

with the real fields ϕ_0, χ_0 and the complex field $\phi_+ = \phi_-^*$. The $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ is broken down to the diagonal $\mathrm{SU}(2)_{L+R}$ symmetry, also known as the custodial symmetry, by the vacuum expectation value v (vev) for the field ϕ_0:

$$\langle 0|H|0\rangle = \frac{v}{\sqrt{2}}.$$

3.3.1 Operators with two-quark and two-lepton fields

The possible operators with two-quark and two-lepton fields can be divided in two groups: the operators with $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$ symmetry and the operator with explicitly broken $\mathrm{SU}(2)_R$. The first group can be set up by considering the helicity combinations of quark and lepton fields, which are four types, namely $(LL)(LL)$, $(LL)(RR)$, $(RR)(LL)$ and $(RR)(RR)$. Additionally we have to consider the flavour diagonal combinations via the isospin matrices τ^a:

$$\mathcal{O}^{(i)}_{LL,LL} = (\bar{Q}_L \Gamma_i Q_L)(L_L \Gamma_i L_L) \tag{3.16}$$
$$\mathcal{P}^{(i)}_{LL,LL} = (\bar{Q}_L \tau^a \Gamma_i Q_L)(L_L \tau^a \Gamma_i L_L) \tag{3.17}$$
$$\mathcal{O}^{(i)}_{LL,RR} = (\bar{Q}_L \Gamma_i Q_L)(L_R \Gamma_i L_R) \tag{3.18}$$
$$\mathcal{O}^{(i)}_{RR,LL} = (\bar{Q}_R \Gamma_i Q_R)(L_L \Gamma_i L_L) \tag{3.19}$$
$$\mathcal{O}^{(i)}_{RR,RR} = (\bar{Q}_R \Gamma_i Q_R)(L_R \Gamma_i L_R) \tag{3.20}$$
$$\mathcal{P}^{(i)}_{RR,RR} = (\bar{Q}_R \tau^a \Gamma_i Q_R)(L_R \tau^a \Gamma_i L_R) \tag{3.21}$$

Please note that helicity combinations like (LR) cannot appear at dimension six, because additional Higgs fields are required for the helicity flip.

The second group form the explicit $\mathrm{SU}(2)_R$ breaking operators, which are constructed by inserting τ^3 into the (RR) parts of the operators discussed above:

$$\mathcal{R}^{(i)}_{LL,RR} = (\bar{Q}_L \Gamma_i Q_L)(L_R \Gamma_i \tau^3 L_R) \tag{3.22}$$
$$\mathcal{R}^{(i)}_{RR,LL} = (\bar{Q}_R \Gamma_i \tau^3 Q_R)(L_L \Gamma_i L_L) \tag{3.23}$$
$$\mathcal{R}^{(i)}_{RR,RR} = (\bar{Q}_R \Gamma_i Q_R)(L_R \Gamma_i \tau^3 L_R) \tag{3.24}$$
$$\mathcal{S}^{(i)}_{RR,RR} = (\bar{Q}_R \tau^a \Gamma_i Q_R)(L_R \tau^a \tau^3 \Gamma_i L_R) \tag{3.25}$$
$$\mathcal{T}^{(i)}_{RR,RR} = (\bar{Q}_R \tau^a \tau^3 \Gamma_i Q_R)(L_R \tau^a \tau^3 \Gamma_i L_R) \tag{3.26}$$

With i we index the Dirac structures encoded in Γ_i:

$$\Gamma_i \otimes \Gamma_i = 1 \otimes 1,\ \gamma_\mu \otimes \gamma^\mu,\ \gamma_\mu \gamma_5 \otimes \gamma_5 \gamma^\mu,\ \sigma_{\mu\nu} \otimes \sigma^{\mu\nu} \tag{3.27}$$

Note that the operators are not all independent. The right-handed neutrino can be integrated out, if we assume it acquires a large Majorana mass, which lies above Λ. Thus all operators right-handed in the lepton part have a projection $(1-\tau^3)/2 = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ involving only the charged leptons like $(\bar{l'}_R \Gamma_i l_R)$. These operators can only appear in neutral currents.

In the decay we are interested in, the $b \to c$ transition is a charged current interaction. Then the leptonic current has to be a charged current, too, meaning an lepton operator involving a charged lepton and a neutral lepton. But since we integrated out the right-handed neutrino, it can only be a left-handed one. Due to the fact that the helicity is conserved in the operators above, the charged lepton has to be left-handed, as well. Finally, only a left-handed leptonic current is possible, yielding only two possible operators in conjunction with the $b \to c$ current:

$$\mathcal{O}_1 = (\bar{b}_L \gamma_\mu c_L)(\bar{\nu}_{\ell,L} \gamma^\mu \ell_L) \quad \mathcal{O}_2 = (\bar{b}_R \gamma_\mu c_R)(\bar{\nu}_{\ell,L} \gamma^\mu \ell_L) \tag{3.28}$$

where $\ell = e, \mu$ or τ.

3.3.2 Operators with two-quark, gauge or Higgs fields

We will group the operators with two quark and gauge and Higgs boson fields of mass dimension six according to the helicity structure of the involved fermion fields, giving an expansion like (3.2):

$$\mathscr{L} = \mathscr{L}_{\text{SM}} + \frac{1}{\Lambda^2}\left(\sum_i \mathcal{O}_{LL}^{(i)} + \sum_i \mathcal{O}_{RR}^{(i)} + \sum_i \mathcal{O}_{LR}^{(i)}\right) \tag{3.29}$$

The two quark fields have dimension three and thus the remaining three dimensions have to come from insertions of covariant derivatives and Higgs fields. We will group the operators according to the helicity structure and consecutively by the combinations of insertions.

LL-Operators

LL-Operators with Three Derivatives The LL-operators with three derivatives are

$$\mathcal{O}_{LL}^{(1)} = G^{(1)}\overline{Q}_L(\mathrm{i}\not{D})^3 Q_L \tag{3.30}$$
$$\mathcal{O}_{LL}^{(2)} = G^{(2)}\overline{Q}_L\left\{\mathrm{i}\not{D},\sigma^{\mu\nu}B_{\mu\nu}\right\}Q_L \tag{3.31}$$
$$\mathcal{O}_{LL}^{(3)} = \mathrm{i}G^{(3)}\overline{Q}_L\left[\mathrm{i}\not{D},\sigma^{\mu\nu}B_{\mu\nu}\right]Q_L \tag{3.32}$$
$$\mathcal{O}_{LL}^{(4)} = G^{(4)}\overline{Q}_L\left\{\mathrm{i}\not{D},\sigma^{\mu\nu}W_{\mu\nu}\right\}Q_L \tag{3.33}$$
$$\mathcal{O}_{LL}^{(5)} = \mathrm{i}G^{(5)}\overline{Q}_L\left[\mathrm{i}\not{D},\sigma^{\mu\nu}W_{\mu\nu}\right]Q_L \tag{3.34}$$
$$\mathcal{O}_{LL}^{(6)} = G^{(6)}\overline{Q}_L\left[\mathrm{i}\mathrm{D}^\mu,\mathrm{i}B_{\mu\nu}\right]\gamma^\nu Q_L \tag{3.35}$$
$$\mathcal{O}_{LL}^{(7)} = \mathrm{i}G^{(7)}\overline{Q}_L\left[\mathrm{i}\mathrm{D}^\mu,\mathrm{i}W_{\mu\nu}\right]\gamma^\nu Q_L. \tag{3.36}$$

The coefficient matrices $G^{(i)}$ have to be hermitian for the operator to be hermitian. The $B_{\mu\nu}$ and $W_{\mu\nu}$ are field strength tensors of the $\mathrm{U}(1)_Y$ and $\mathrm{SU}(2)_L$ symmetries respectively. The field strength tensors can be written as commutators of covariant derivatives and thus count as two derivatives. They have to be rated separately, because the aforementioned commutators give only linear combinations of them.

LL-Operators with two Higgs fields and one Derivative The LL-operators with two Higgs fields and one derivative are

$$\mathcal{O}_{LL}^{(8)} = \overline{Q}_L\left\{H\hat{G}^{(8)}H^\dagger,\mathrm{i}\not{D}\right\}Q_L \tag{3.37}$$
$$\mathcal{O}_{LL}^{(9)} = \overline{Q}_L\left[H\hat{G}^{(9)}H^\dagger,\mathrm{i}\not{D}\right]Q_L \tag{3.38}$$
$$\mathcal{O}_{LL}^{(10)} = \overline{Q}_L H\hat{G}^{(10)}(\mathrm{i}\not{D})H^\dagger Q_L. \tag{3.39}$$

Again the matrices $\hat{G}^{(i)}$ are hermitian. Due to the helicity flips via the Higgs fields here it is possible to have explicitly $\mathrm{SU}(2)_R$ breaking terms, reflected by the definition of the matrices $\hat{G}^{(i)}$, which consist of a $\mathrm{SU}(2)_R$ conserving and a $\mathrm{SU}(2)_R$ breaking part:

$$\hat{G}^{(i)} = G^{(i)} + G^{(i)\prime}\tau_3. \tag{3.40}$$

RR-Operators

RR-Operators with three derivatives

$$\mathcal{O}_{RR}^{(1)} = F^{(1)}\overline{Q}_R(\mathrm{i}\not{D})^3 Q_R \tag{3.41}$$
$$\mathcal{O}_{RR}^{(2)} = F^{(2)}\overline{Q}_R\left\{\mathrm{i}\not{D},\sigma^{\mu\nu}B_{\mu\nu}\right\}Q_R \tag{3.42}$$
$$\mathcal{O}_{RR}^{(3)} = F^{(3)}\overline{Q}_R\left[\mathrm{i}\not{D},\sigma^{\mu\nu}B_{\mu\nu}\right]Q_R \tag{3.43}$$
$$\mathcal{O}_{RR}^{(4)} = F^{(4)}\overline{Q}_R\left[\mathrm{iD}^{\mu}, iB_{\mu\nu}\right]\gamma^{\nu}Q_R \tag{3.44}$$

Additional operators explicitly breaking the $\mathrm{SU}(2)_R$ symmetry are not displayed, because they can be obtained by replacing all Q_R by $\tau^3 Q_R$ in all possible ways.

RR-Operators with two Higgs fields and one derivative

$$\mathcal{O}_{RR}^{(5)} = F^{(5)}\overline{Q}_R\left\{H^{\dagger}H,\mathrm{i}\not{D}\right\}Q_R \tag{3.45}$$
$$\mathcal{O}_{RR}^{(6)} = \mathrm{i}F^{(6)}\overline{Q}_R\left[H^{\dagger}H,\mathrm{i}\not{D}\right]Q_R \tag{3.46}$$
$$\mathcal{O}_{RR}^{(7)} = F^{(7)}\overline{Q}_R H^{\dagger}(\mathrm{i}\not{D})HQ_R \tag{3.47}$$
$$\tag{3.48}$$

Again additional operators explicitly breaking the $\mathrm{SU}(2)_R$ symmetry can be obtained by replacing all Q_R by $\tau^3 Q_R$ in all possible ways.

LR-Operators

LR-Operators with one Higgs field and two derivatives

$$\mathcal{O}_{LR}^{(1)} = K^{(1)}\overline{Q}_L HH^{\dagger}HQ_R + \mathrm{h.c.} \tag{3.49}$$

Due to the added hermitian conjugate of the first part the matrix $K^{(1)}$ needs not to be hermitian. A corresponding $\mathrm{SU}(2)_R$ operator can again be obtained by replacing all Q_R by $\tau^3 Q_R$.

LR-Operators with one Higgs field and two Derivatives

$$\mathcal{O}_{LR}^{(2)} = \overline{Q}_L H\hat{K}^{(2)}(\mathrm{i}\not{D})^2 Q_R + \mathrm{h.c.} \tag{3.50}$$
$$\mathcal{O}_{LR}^{(3)} = \overline{Q}_L(\mathrm{i}\not{D})^2 H\hat{K}^{(3)}Q_R + \mathrm{h.c.} \tag{3.51}$$
$$\mathcal{O}_{LR}^{(4)} = \overline{Q}_L\sigma^{\mu\nu}B_{\mu\nu}H\hat{K}^{(4)}Q_R + \mathrm{h.c.} \tag{3.52}$$
$$\mathcal{O}_{LR}^{(5)} = \overline{Q}_L\sigma^{\mu\nu}W_{\mu\nu}H\hat{K}^{(5)}Q_R + \mathrm{h.c.} \tag{3.53}$$
$$\mathcal{O}_{LR}^{(6)} = \overline{Q}_L(\mathrm{i}\not{D})H\hat{K}^{(6)}(\mathrm{i}\not{D})Q_R + \mathrm{h.c.} \tag{3.54}$$
$$\mathcal{O}_{LR}^{(7)} = \overline{Q}_L(\mathrm{iD}_{\mu})H\hat{K}^{(7)}(\mathrm{iD}^{\mu})Q_R + \mathrm{h.c.} \tag{3.55}$$

The matrices $\hat{K}^{(2)}$ consist of a $\mathrm{SU}(2)_R$ conserving and a $\mathrm{SU}(2)_R$ breaking part:

$$\hat{K}^{(i)} = K^{(i)} + K^{(i)\prime}\tau_3. \tag{3.56}$$

Both $K^{(i)}$ and $K^{(i)\prime}$ need not to be hermitian.

Reduction of the Number of Operators

We discuss possible reductions of the number of operators. As noted in the beginning it is important to have a limited number of parameters for parametrizing possible new physics effects, which is directly related to the amount of operators. Fortunately, the operators above are not all independent. Some of them are connected by the equation of motion. This allows us to reduce their number considerably in a most general manner. In a second step we will apply the operator basis to the considered decay $b \to c l \nu$, which finally ends up in a simple enhancement of the $b \to c$ current with only six parameters.

The equation of motions for the involved fields are

$$(\mathrm{i}\not{D})Q_L = \frac{1}{v} H \mathcal{M} Q_R \tag{3.57}$$

$$(\mathrm{i}\not{D})Q_R = \frac{1}{v} H \mathcal{M}^\dagger Q_L. \tag{3.58}$$

This connects the operators with three covariant derivatives, except for $\mathcal{O}_{LL}^{(6)}$ and $\mathcal{O}_{LL}^{(7)}$, with the operators with two derivatives and one Higgs field of which we eliminate the first. The operators $\mathcal{O}_{LL}^{(6)}$ and $\mathcal{O}_{LL}^{(7)}$ on the other hand can be rewritten by the equation of motion for the gauge fields and turn into four-fermion operators which we discussed in the previous section.

We can also remove all operators which after spontaneous symmetry breaking contribute to operators of the kinetic energy. These contributions correspond only to field redefinition, mass renormalization and renormalization of the CKM matrix.

Basis of Operators

After application of the equations of motion and the kinetic energy type operators we have the following operator basis, more explicitly showing displaying the $\mathrm{SU}(2)_R$ breaking operators:

LL-Operators

$$O_{LL}^{(1)} = \bar{Q}_L \not{L}\, Q_L \tag{3.59}$$

$$O_{LL}^{(2)} = \bar{Q}_L \not{L}_3\, Q_L \tag{3.60}$$

with

$$L^\mu = H \left(iD^\mu H\right)^\dagger + \left(iD^\mu H\right) H^\dagger \tag{3.61}$$

$$L_3^\mu = H \tau_3 \left(iD^\mu H\right)^\dagger + \left(iD^\mu H\right) \tau_3 H^\dagger \tag{3.62}$$

RR-Operators

$$O_{RR}^{(1)} = \bar{Q}_R \not{R}\, Q_R \tag{3.63}$$

$$O_{RR}^{(2)} = \bar{Q}_R \left\{\tau_3, \not{R}\right\} Q_R \tag{3.64}$$

$$O_{RR}^{(3)} = i\bar{Q}_R \left[\tau_3, \not{R}\right] Q_R \tag{3.65}$$

$$O_{RR}^{(4)} = \bar{Q}_R\, \tau_3 \not{R} \tau_3\, Q_R \tag{3.66}$$

with

$$R^\mu = H^\dagger \left(iD^\mu H\right) + \left(iD^\mu H\right)^\dagger H \tag{3.67}$$

LR-Operators

$$O_{LR}^{(1)} = \bar{Q}_L \left(\sigma_{\mu\nu} B^{\mu\nu}\right) H\, Q_R + \text{h.c.} \tag{3.68}$$

$$O_{LR}^{(2)} = \bar{Q}_L \left(\sigma_{\mu\nu} W^{\mu\nu}\right) H\, Q_R + \text{h.c.} \tag{3.69}$$

$$O_{LR}^{(3)} = \bar{Q}_L \left(iD_\mu H\right) iD^\mu\, Q_R + \text{h.c.} \tag{3.70}$$

3.3.3 Application to the $b \to cl\nu$ transition

We consider the contributions of the operators to the $b \to c$ transition after spontaneous symmetry breaking. This selects only the $b \to c$ part of the doublet combinations with dimension 3 for both fields, the covariant derivative gives the gauge boson interaction with mass dimension 1, and the vacuum expectation value squared v^2 from spontaneous symmetry breaking yields 2 mass dimensions, summing up to the required mass dimension of 6.

$$O_{LL}^{(1)} = \frac{v^2 g}{\sqrt{2}} G_{cb}^{(1)} V_{cb}\, \bar{c} \not{W}^+ P_- b \tag{3.71}$$

$$O_{RR}^{(1)} = \frac{v^2 g}{\sqrt{2}} F_{cb}^{(1)} V_{cb}\, \bar{c} \not{W}^+ P_+ b \tag{3.72}$$

$$O_{RR}^{(3)} = \frac{v^2 g}{\sqrt{2}} 2iF_{cb}^{(3)} V_{cb}\, \bar{c} \not{W}^+ P_+ b \tag{3.73}$$

$$O_{RR}^{(4)} = -\frac{v^2 g}{\sqrt{2}} F_{cb}^{(4)} V_{cb}\, \bar{c} \not{W}^+ P_+ b \tag{3.74}$$

$$O_{LR}^{(3)} = \frac{vg}{2} V_{cb}\, \bar{c} \sigma^{\mu\nu} \left\{ \partial_\mu W_\nu^+ \widetilde{K}_{cb}^{(3)} P_+ + \partial_\nu W_\mu^+ \widetilde{K}_{cb}^{\dagger(3)} P_- \right\} b \tag{3.75}$$

$$O_{LR}^{(4)} = \frac{vg}{2} V_{cb}\, \bar{c} \left\{ W_\mu^+ i\partial^\mu \widetilde{K}_{cb}^{(4)} P_+ + W_\mu^+ i\partial^\mu \widetilde{K}_{cb}^{\dagger(4)} P_- \right\} b \tag{3.76}$$

with the left- and right-handed projectors $P_\pm = (1 \pm \gamma_5)/2$ and the couplings $\widetilde{K}_{cb}^{(i)} = K_{cb}^{(i)} - K_{cb}^{\prime(i)}$

We have discussed so far the contributions from dimension six operators. For a combined analysis we have to include also the dimension four standard model contribution

$$\mathcal{L}_{SM} = \frac{g}{\sqrt{2}} V_{cb}\, \bar{c} \not{W}^+ P_- b. \tag{3.77}$$

The Lagrangian with the standard-model contribution and the generalized dimension six interaction becomes

$$\begin{aligned}
\mathcal{L} = &\frac{g}{\sqrt{2}} \left(1 + \frac{v^2}{\Lambda^2} G_{cb}^{(1)}\right) V_{cb}\, \bar{c} \not{W}^+ P_- b \\
&+ \frac{v^2 g}{\sqrt{2}} \frac{1}{\Lambda^2} \left(F_{cb}^{(1)} - F_{cb}^{(4)} + 2iF_{cb}^{(3)}\right) V_{cb}\, \bar{c} \not{W}^+ P_+ b \\
&+ \frac{vg}{2} \frac{1}{\Lambda^2} V_{cb}\, \bar{c} \left\{ W_\mu^+ i\partial^\mu \widetilde{K}_{cb}^{(4)} P_+ + W_\mu^+ i\partial^\mu \widetilde{K}_{cb}^{\dagger(4)} P_- \right\} b \\
&+ \frac{vg}{2} \frac{1}{\Lambda^2} V_{cb}\, \bar{c} \sigma^{\mu\nu} \left\{ \partial_\mu W_\nu^+ \widetilde{K}_{cb}^{(3)} P_+ + \partial_\nu W_\mu^+ \widetilde{K}_{cb}^{\dagger(3)} P_- \right\} b.
\end{aligned} \tag{3.78}$$

Turning to the energy scale of the considered decay (hadronic energy scale) we can integrate out the weak boson similar to the field component H_v of the heavy degrees of freedom in the introduction to the heavy-quark effective theory. Because the weak boson connects the quark current with the leptonic current, integrating out the weak bosons amount for the replacement

$$W_\mu^- = \frac{4G_F}{\sqrt{2}}\bar{e}\gamma_\mu P_- \nu_e. \tag{3.79}$$

resulting in the effective Hamiltonian

$$\mathscr{H}_{\text{eff}} = \frac{4G_F V_{cb}}{\sqrt{2}} J_{q,\mu} J_l^\mu, \tag{3.80}$$

where $J_l^\mu = \bar{e}\,\gamma^\mu P_-\,\nu_e$ is the left-handed leptonic current and $J_{h,\mu}$ is the generalized hadronic $b \to c$ given by

$$\begin{aligned} J_{h,\mu} = {} & c_L\ \bar{c}\gamma_\mu P_- b + c_R\ \bar{c}\gamma_\mu P_+ b + g_L\ \bar{c}\, i\overleftrightarrow{D}_\mu P_- b + g_R\ \bar{c}\, i\overleftrightarrow{D}_\mu P_+ b \\ & + d_L\ i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_- b) + d_R\ i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_+ b)\,, \end{aligned} \tag{3.81}$$

where we have renamed the coupling constants to streamline the notation. The term proportional to c_L contains the standard model contribution and a dimension six contribution. Including the result from the discussion of the two-quark-two-leptons operators we find only additional contributions to $c_{\rm L}$ and $c_{\rm R}$.

The enhanced current (3.81) could have been an ansatz for non-standard-model contributions from the start, but using an effective field theory approach unveils the orders of magnitude of the coefficients. The derivatives in the (3.78) yield the momenta of the involved particles and hence we have to assign the typical scale for the derivatives to be at the mass of the b quark. This has dispose us to include the inverse b quark mass in the couplings with the coefficients $g_{L/R}$ and $d_{L/R}$.

To estimate the orders of magnitude of the coefficients we compare (3.81) with (3.78):

$$c_L \propto 1; \qquad c_R \propto \frac{v^2}{\Lambda^2}; \qquad d_{R/L} \propto \frac{v\, m_b}{\Lambda^2}; \qquad g_{R/L} \propto \frac{v\, m_b}{\Lambda^2}; \tag{3.82}$$

We note further that in minimal flavour violating scenarios [24] any occurrence of a right handed quark is related to a helicity flip and hence mass factors occur. While the above estimates remain the same for $g_{R/L}$ and $d_{R/L}$, the estimate for c_R contains a strong additional suppression factor of $m_b m_c/v^2$. The additional factor m_b/Λ in $g_{L/R}$ and $d_{L/R}$ reflects the fact that in order to obtain a helicity flip one has to have an additional Yukawa coupling, making these contributions small compared to the helicity-conserving ones.

Due to squaring the amplitude for the calculation of the rate or moments an interference term of the non-standard coupling with the standard model part appears besides the new contributions squared. The relevant part is the interference contribution because of the smallness of the parameters. Occurring couplings between left and right handed parts in these interference terms require a helicity flip of the final state c quark, adding a further suppression factor of $m_{\rm c}/m_{\rm b}$.

The orders for the contributions in the interference terms are thus for the rates we obtain additional contributions of the orders

$$c_L c_R \sim \frac{m_c}{m_b}\frac{v^2}{\Lambda^2}, \qquad c_L d_L \sim c_L g_L \sim \frac{m_b v}{\Lambda^2} \quad \text{and} \quad c_L d_R \sim c_L g_R \sim \frac{m_c}{m_b}\frac{m_b v}{\Lambda^2}. \tag{3.83}$$

4 Power Corrections to the Inclusive Semi-leptonic Decay $\overline{B} \rightarrow X_c \ell \overline{\nu}_\ell$

4.1 Operator Product Expansion

For the calculation of the inclusive semi-leptonic decay $\overline{B} \to X_c \ell \overline{\nu}_\ell$ in the fashion of the heavy-quark expansion (HQE) it is convenient to express the differential decay rate as

$$\mathrm{d}\Gamma = \frac{\mathrm{G}_\mathrm{F}^2}{m_\mathrm{B}} |V_{cb}|^2 \,\mathrm{Im}\, T_{\mu\nu}\, L^{\mu\nu}\ \mathrm{d}\Pi \tag{4.1}$$

with the hadronic tensor $T_{\mu\nu}$, the leptonic tensor $L^{\mu\nu}$ and the phase space element $\mathrm{d}\Pi$.

The leptonic tensor is

$$\begin{aligned} L^{\mu\nu} =& \sum_{\text{spins}} \langle 0 | J_l^\mu(x) J_l^\nu(0) | 0 \rangle = \sum_{\text{spins}} \langle 0 | \bar{e}(x) \gamma^\mu P_L \nu(x) \bar{\nu}(0) \gamma^\nu P_L e(0) | 0 \rangle \\ =& \mathrm{Tr} \left[\not{p}_e \gamma^\mu P_L \not{p}_\nu \gamma^\nu P_L \right] \\ =& 2 \left(p_e^\mu p_\nu^\nu - (p_e \cdot p_\nu) g^{\mu\nu} + p_e^\nu p_\nu^\mu - i \epsilon^{\alpha\nu\beta\mu} p_{e,\alpha} p_{\nu,\beta} \right) \end{aligned} \tag{4.2}$$

and the hadronic tensor describes a forward scattering amplitude (see fig. 4.1):

$$T_{\mu\nu} = \int \mathrm{d}^4 x\, e^{-\mathrm{i}x(m_\mathrm{b} v - q)} \langle \bar{B}(p) |\, \mathrm{T} \left[\bar{b}_v(x) \Gamma_\mu\, c(x) \bar{c}(0) \Gamma_\nu^\dagger\, b_v(0) \right] | \bar{B}(p) \rangle. \tag{4.3}$$

where $\bar{b}_v(x)\Gamma_\mu\, c(x)$ and $\bar{c}(0)\Gamma_\nu^\dagger\, b_v$ refer to the enhanced $b \to c$ current from (3.81). An exhaustive derivation of (4.1) with (4.3) can be found in [25] and [26]. The following derivations are described along the lines of [20] and [27]. The contraction of $c(x)\bar{c}(0)$ gives the propagator of the charm quark in the background field of soft gluons in the B meson. The momentum of the charm quark is $p_\mathrm{c} = p_\mathrm{b} - q = m_\mathrm{b} v + k - q$. Thus the background field propagator is according to the correspondence $k \leftrightarrow \mathrm{i}\mathrm{D}$:

$$\mathrm{i} S_\mathrm{BGF} = \frac{1}{\not{Q} + \mathrm{i}\not{D} - m_\mathrm{c}} \tag{4.4}$$

with $Q = m_\mathrm{b} v - q$.

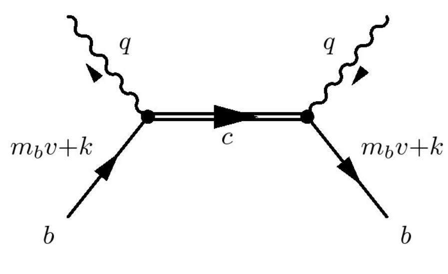


Figure 4.1: *Forward scattering of the bottom quark*

The OPE of the hadronic tensor is performed by the expansion of the background field propagator in the covariant derivatives iD [1]:

$$
\begin{aligned}
iS_{\mathrm{BGF}} =& \frac{1}{\not{Q} - m_\mathrm{c}} - \frac{1}{\not{Q} - m_\mathrm{c}} i\not{D} \frac{1}{\not{Q} - m_\mathrm{c}} + \frac{1}{\not{Q} - m_\mathrm{c}} i\not{D} \frac{1}{\not{Q} - m_\mathrm{c}} i\not{D} \frac{1}{\not{Q} - m_\mathrm{c}} \\
& - \frac{1}{\not{Q} - m_\mathrm{c}} i\not{D} \frac{1}{\not{Q} - m_\mathrm{c}} i\not{D} \frac{1}{\not{Q} - m_\mathrm{c}} i\not{D} \frac{1}{\not{Q} - m_\mathrm{c}} + \ldots
\end{aligned} \tag{4.5}
$$

The displayed terms are the ones needed for the calculation of the $\mathcal{O}(1/m_\mathrm{b}^3)$ in the HQE.

The field b_v is the full QCD field but redefined by the phase factor $e^{im_\mathrm{b} v \cdot x}$ to remove the large part of the b-quark momentum:

$$b_v(x) = e^{im_\mathrm{b} v \cdot x} b(x).$$

With the help of the Dirac equation for the redefined field b_v

$$0 = (\mathrm{i}\not{D} - m_\mathrm{b}) b(x) = (\mathrm{i}\not{D} - m_\mathrm{b}) e^{-im_\mathrm{b} v \cdot x} b_v(x) = e^{-im_\mathrm{b} v \cdot x} (\mathrm{i}\not{D} + m_\mathrm{b} \not{v} - m_\mathrm{b}) b_v(x) \tag{4.6}$$

$$\Rightarrow \quad (\mathrm{i}\not{D} + m_\mathrm{b} \not{v} - m_\mathrm{b}) b_v(x) = 0 \tag{4.7}$$

the following useful relations for b_v can be derived:

$$\not{v} b_v = b_v - \frac{1}{m_\mathrm{b}} \mathrm{i}\not{D} b_v \tag{4.8}$$

$$P_+ b_v = b_v - \frac{1}{2m_\mathrm{b}} \mathrm{i}\not{D} b_v \tag{4.9}$$

$$P_- b_v = \frac{1}{2m_\mathrm{b}} \mathrm{i}\not{D} b_v \tag{4.10}$$

$$(\mathrm{i}v \cdot \mathrm{D}) b_v = -\frac{1}{2m_\mathrm{b}} i\not{D} \mathrm{i}\not{D} b_v \tag{4.11}$$

with $P_\pm = (1 \pm \not{v})/2$ being the projectors on the heavy and light components of b_v.

4.2 Trace Formulas

In the further calculation we will need only operators of dimension 5, but in the following considerations we will include operators of dimension 6 for didactical reasons:

dimension	operator
3	$\bar{b}_v \Gamma b_v$
4	$\bar{b}_v (\mathrm{iD}_\alpha) \Gamma b_v$
5	$\bar{b}_v (\mathrm{iD}_\alpha)(\mathrm{iD}_\beta) \Gamma b_v$
6	$\bar{b}_v (\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)(\mathrm{iD}_\gamma) \Gamma b_v$

(4.12)

[1] This is done by applying the operator identity

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A} B \frac{1}{A+B}$$

recursively.

with Γ being an arbitrary Dirac matrix. To parametrize these operators they could be expanded in the basis of 16 Dirac matrices, which are 1, γ_5, $\gamma_\mu\gamma_5$ and $\sigma_{\mu\nu}$. But considering the operator with the highest dimension in analysis to perform, we can reduce the number of occurring Dirac structures. If we assume the highest dimensional operator not to have a $1/m_\mathrm{b}$ correction then the relations (4.8) to (4.11) become

$$\not{v} b_v = b_v \tag{4.13}$$

$$P_+ b_v = b_v \tag{4.14}$$

$$P_- b_v = 0 \tag{4.15}$$

$$(\mathrm{i}v \cdot \mathrm{D}) b_v = 0. \tag{4.16}$$

Hence at that order the light component of the field vanishes. As the heavy $e^{im_\mathrm{b} v\cdot x}$ part has been removed from the field, there is no time dependence left, meaning b_v is a static field in the case of the highest dimension in the performed $1/m_\mathrm{b}$ expansion. This is in accordance with the derivation of the HQET where we showed that the heavy degrees of freedom are integrated out and thus become static.

In this case the field b_v carries implicitly the projector P_+ because of (4.14) and hence the Dirac matrix Γ is sandwiched between two projectors $P_+\Gamma P_+$, which amounts for the following replacement after expanding Γ to the basic Dirac matrices:

$$1 \to P_+ \tag{4.17}$$

$$\gamma_5 \to 0 \tag{4.18}$$

$$\gamma_\mu \to v_\mu P_+ \tag{4.19}$$

$$\gamma_\mu\gamma_5 \to s_\mu \tag{4.20}$$

$$-\mathrm{i}\sigma_{\mu\nu} \to \mathrm{i}v^\alpha \epsilon_{\alpha\mu\nu\beta} s^\beta \tag{4.21}$$

where we defined the spin matrices $s_\mu = P_+\gamma_\mu\gamma_5 P_+$. They are a generalization of the Pauli spin matrices for the frame moving with velocity v. The spin matrices satisfy the relations

$$s_\mu s_\nu = (-g_{\mu\nu} + v_\mu v_\nu)P_+ + \mathrm{i}\epsilon_{\alpha\mu\nu\beta} v^\alpha s^\beta \qquad v \cdot s = 0. \tag{4.22}$$

With the replacements (4.17) to (4.21) and the relations (4.22) the Dirac matrix Γ sandwiched between the projectors can be expanded as

$$P_+\Gamma P_+ = \frac{1}{2} P_+ \operatorname{Tr} P_+\Gamma - \frac{1}{2} s_\nu \operatorname{Tr} s^\nu \Gamma. \tag{4.23}$$

Thus only two operators are needed to built up all Dirac structures occurring in the static case, namely

$$\mathcal{O}^{(1)}_{\alpha\beta\gamma} = \bar{b}_v (\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)(\mathrm{iD}_\gamma)\Gamma b_v \tag{4.24}$$

$$\mathcal{O}^{(s)}_{\alpha\beta\gamma\lambda} = \bar{b}_v (\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)(\mathrm{iD}_\gamma) s_\lambda \Gamma b_v \tag{4.25}$$

or because of the linear dependence of $\sigma_{\mu\nu}$ and s_λ:

$$\mathcal{O}^{(1)}_{\alpha\beta\gamma} = \bar{b}_v (\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)(\mathrm{iD}_\gamma)\Gamma b_v \tag{4.26}$$

$$\mathcal{O}^{(s)}_{\alpha\beta\gamma\mu\nu} = \bar{b}_v (\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)(\mathrm{iD}_\gamma)(-\mathrm{i}\sigma_{\mu\nu})\Gamma b_v. \tag{4.27}$$

Contracting the Lorentz indices with the allowed Lorentz structures $g^{\alpha\beta}$, $g^{\alpha\gamma}$, $g^{\beta\gamma}$, v^α, v^β and v^γ the so-called basic scalar parameters at the highest order investigated can be derived. Due to the fields b_v being static and (4.16) the contraction of the first and the last covariant derivative with the velocity v vanishes. A the order $1/m_\mathrm{b}^3$ there are two basic parameters, the Darwin term $\hat{\rho}_D$ and the spin-orbit term $\hat{\rho}_\mathrm{LS}$ defined by

$$2m_\mathrm{B}\hat{\rho}_D^3 =\langle\bar{B}(p)|\bar{b}_v(\mathrm{iD}_\mu)(\mathrm{i}v\cdot\mathrm{D})(\mathrm{iD}^\mu)b_v|\bar{B}(p)\rangle \tag{4.28}$$
$$2m_\mathrm{B}\hat{\rho}_\mathrm{LS}^3 =\langle\bar{B}(p)|\bar{b}_v(\mathrm{iD}_\mu)(\mathrm{i}v\cdot\mathrm{D})(\mathrm{iD}_\nu)(-\mathrm{i}\sigma^{\mu\nu})b_v|\bar{B}(p)\rangle \tag{4.29}$$

Assuming the order $1/m_\mathrm{b}^2$ to be static, also two parameters can be found, namely the kinetic energy parameter $\hat{\mu}_\pi$ and the chromomagnetic moment $\hat{\mu}_G$

$$2m_\mathrm{B}\hat{\mu}_\pi^2 =-\langle\bar{B}(p)|\bar{b}_v(\mathrm{iD})^2b_v|\bar{B}(p)\rangle \tag{4.30}$$
$$2m_\mathrm{B}\hat{\mu}_G^2 =\langle\bar{B}(p)|\bar{b}_v(\mathrm{iD}_\mu)(\mathrm{iD}_\nu)(-\mathrm{i}\sigma^{\mu\nu})b_v|\bar{B}(p)\rangle. \tag{4.31}$$

At order $1/m_\mathrm{b}$ there exists no new basic parameters, because all matrix elements involving one covariant derivative can be related to higher dimensional matrix elements by the equation of motion. At parton level ($\mathcal{O}(1/m_\mathrm{b}^0)$) we have only one parameter, the normalization of the state to the B meson mass m_B:

$$2m_\mathrm{B}=\langle\bar{B}(p)|\bar{b}_v\not{v}b_v|\bar{B}(p)\rangle. \tag{4.32}$$

For the considerations at hand the highest order is $1/m_\mathrm{b}^3$ and the lower orders cannot be treated as being static. To evaluate the needed matrix at the lower orders we have to take into account all possible Dirac structures:

$$\langle\bar{B}(p)|\bar{b}_{v,\alpha}(\mathrm{iD}_\mu)(\mathrm{iD}_\nu)\Gamma b_{v,\beta}|\bar{B}(p)\rangle=\sum_i\Gamma\,\hat{\Gamma}^{(i)}_{\alpha\beta}A^{(i)}_{\mu\nu} \tag{4.33}$$

where $\hat{\Gamma}^{(i)}$ are the 16 basic Dirac matrices. Note that we have explicitly shown the spinor indices, indicating that the spin sum is not yet performed. This is obvious as the right-hand side contains Dirac matrices. Again the tensors $A^{(i)}_{\mu\nu}$ can be related to the basic parameters via contracting both sides with all allowed Lorentz structures, but this time b_v is not in the static limit and thus the relations (4.8) to (4.11) apply connecting the matrix elements with the higher order in the $1/m_\mathrm{b}$ expansion. The tensor $A^{(i)}_{\mu\nu}$ is then expressed in basic parameters of $1/m_\mathrm{b}^2$ and $1/m_\mathrm{b}^3$.

The evaluation of the matrix elements of lower dimensions can now be calculated recursively starting from the operator of highest dimension as shown above. Once all tensors $A^{(i)}$ of all orders are calculated we can express the hadronic tensor in the basic parameters as a so-called trace formula:

$$T_{\mu\nu}=\langle\bar{B}(p)|\mathrm{T}\left[\bar{b}_v\Gamma_\mu S_\mathrm{BGF}\Gamma_\nu^\dagger b_v\right]|\bar{B}(p)\rangle \tag{4.34}$$
$$=\sum_i\mathrm{Tr}\left\{\Gamma_\mu\frac{1}{\not{Q}-m_\mathrm{c}}\Gamma_\nu^\dagger\hat{\Gamma}^{(i)}\right\}A^{(i,0)} \tag{4.35}$$
$$+\sum_i\mathrm{Tr}\left\{\Gamma_\mu\frac{1}{\not{Q}-m_\mathrm{c}}\gamma^\alpha\frac{1}{\not{Q}-m_\mathrm{c}}\Gamma_\nu^\dagger\hat{\Gamma}^{(i)}\right\}A^{(i,1)}_\alpha \tag{4.36}$$
$$+\sum_i\mathrm{Tr}\left\{\Gamma_\mu\frac{1}{\not{Q}-m_\mathrm{c}}\gamma^\alpha\frac{1}{\not{Q}-m_\mathrm{c}}\gamma^\beta\frac{1}{\not{Q}-m_\mathrm{c}}\Gamma_\nu^\dagger\hat{\Gamma}^{(i)}\right\}A^{(i,2)}_{\alpha\beta} \tag{4.37}$$
$$+\sum_i\mathrm{Tr}\left\{\Gamma_\mu\frac{1}{\not{Q}-m_\mathrm{c}}\gamma^\alpha\frac{1}{\not{Q}-m_\mathrm{c}}\gamma^\beta\frac{1}{\not{Q}-m_\mathrm{c}}\gamma^\gamma\frac{1}{\not{Q}-m_\mathrm{c}}\Gamma_\nu^\dagger\hat{\Gamma}^{(i)}\right\}A^{(i,3)}_{\alpha\beta\gamma} \tag{4.38}$$

The expansion of the background field propagator yields the correct ordering of the covariant derivatives, hence a separate calculation of the gluon-matrix elements as in standard calculations (e. g. [26]) is not needed.

The relevant matrix elements for our considerations in terms of the basic parameters are

$$
\begin{aligned}
\langle \bar{B}(p)|\bar{b}_v(\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)(\mathrm{iD}_\gamma)b_v|B(p)\rangle &= \frac{m_\mathrm{B}}{3}v^\beta P_+\left[(g^{\alpha\gamma}-v^\alpha v^\gamma)\,\hat{\rho}_\mathrm{D}^3+\frac{1}{2}\mathrm{i}\sigma^{\alpha\gamma}\,\hat{\rho}_\mathrm{LS}^3\right]\\
\langle \bar{B}(p)|\bar{b}_v(\mathrm{iD}_\alpha)(\mathrm{iD}_\beta)b_v|B(p)\rangle &= \frac{m_\mathrm{B}}{3}P_+\left[(g^{\alpha\beta}-v^\alpha v^\beta)(-\,\hat{\mu}_\pi^2)+\frac{1}{2}\mathrm{i}\sigma^{\alpha\beta}\,\hat{\mu}_\mathrm{G}^2\right]\\
&\quad+\frac{m_\mathrm{B}}{6m_b}\left(P_+v^\alpha\gamma^\beta+P_-\gamma^\alpha v^\beta+4P_+\mathrm{i}\sigma^{\alpha\beta}\right)(\hat{\rho}_\mathrm{D}^3+\hat{\rho}_\mathrm{LS}^3)\\
\langle \bar{B}(p)|\bar{b}_v(\mathrm{iD}_\alpha)b_v|B(p)\rangle &= -\frac{m_\mathrm{B}}{2m_\mathrm{b}}\left(P_+v^\alpha-\frac{1}{3}(\gamma^\alpha-\gamma^\alpha\not{v})(\hat{\mu}_\mathrm{G}^2-\hat{\mu}_\pi^2)\right)\\
&\quad+\frac{m_\mathrm{B}}{12m_\mathrm{b}^2}\left(\gamma^\alpha-4v^\alpha\not{v}\right)(\hat{\rho}_\mathrm{D}^3+\hat{\rho}_\mathrm{LS}^3)\\
\langle \bar{B}(p)|\bar{b}_v b_v|B(p)\rangle &= P_+m_\mathrm{B}+\frac{m_\mathrm{B}}{4m_\mathrm{b}^2}(\hat{\mu}_\mathrm{G}^2-\hat{\mu}_\pi^2)
\end{aligned}
$$

The matrix elements summarize very nicely our discussion about the basic parameters and the connection to higher dimensions. The expressions for the matrix elements with three and two covariant derivatives show terms not suppressed my m_b, which represent the parametrization in the static limit. The spin independent term and the spin dependent term of both expressions have the same structure. In the case of the matrix element with two covariant derivatives we are not in the static limit and terms with parameters from higher dimensional operators appear, but suppressed by m_b, which is obvious because of (4.8) to (4.11).

In our analysis we will only perform the HQE up to $\mathcal{O}(1/m_\mathrm{b}^2)$. The expansion (4.5) will be applied only to second order in iD and in the trace formulas we have to neglect the terms involving $\hat{\rho}_\mathrm{D}^3$ and $\hat{\rho}_\mathrm{LS}^3$.

4.3 Imaginary Part

The hadronic tensor $T_{\mu\nu}$ describing the forward scattering of a B meson is related to the hadronic tensor describing the inclusive decay [25, 26]:

$$
W_{\mu\nu}=\sum_{X_c}\langle\bar{B}(p_B)|J_{q,\mu}^\dagger|X_c(p_{X_c})\rangle\langle X_c(p_{X_c})|J_{q,\nu}|\bar{B}(p_B)\rangle(2\pi)^3\delta^4\big(p_B-p_e-p_\nu-p_{X_c}\big). \tag{4.39}
$$

by taking the imaginary part:

$$
-\frac{1}{\pi}\,\mathrm{Im}\,T_{\mu\nu}=W_{\mu\nu} \tag{4.40}
$$

This corresponds to cuts of propagators in the Feynman diagram of the forward scattering amplitude. This in turn means putting the propagating particle on its mass shell by the following relation:

$$
-\frac{1}{\pi}\,\mathrm{Im}\,\frac{1}{p^2-m^2+i\epsilon}=\delta\big(p^2-m^2\big).
$$

In the heavy-quark expansion there is only the charm quark propagator in the sense of the expanded background field propagator S_BGF to cut. Higher orders of the denominator yield derivatives of the delta function:

$$
-\frac{1}{\pi}\,\mathrm{Im}\left(\frac{1}{(m_\mathrm{b}v-q)^2-m_\mathrm{c}^2+i\epsilon}\right)^{n+1}=\frac{(-1)^n}{n!}\delta^{(n)}\big((m_\mathrm{b}v-q)^2-m_\mathrm{c}^2\big).
$$

The derivatives have to be taken off the delta function by partial integration.

4.4 Phase Space Parametrization

The phase space of this three body decay is determined by *two* independent parameters in the rest frame of the decaying particle. This can be deduced from the following consideration: the three momenta of the final state particles give $3 \times 3 = 9$ parameters. The energy and momentum conservation constraints the parameters by four equations and thus $9 - 4 = 5$ degrees of freedom are left. The arbitrary choice of the rotation of the coordinate axis in the rest frame reduces the number by 3 and we end up with the *two* parameters mentioned above.

In the rest frame of the decaying b-quark we choose to align the z-axis with the charged lepton momentum p_l. The antineutrino momentum p_v can be parametrized with one angle relative to the z-axis by placing it in the x-z-plane and the c-quark momentum p_c is determined by four-momentum conservation:

$$p_\mathrm{b}^\mu = \begin{pmatrix} m_\mathrm{b} \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad p_\mathrm{l}^\mu = \begin{pmatrix} E_\mathrm{l} \\ 0 \\ 0 \\ E_\mathrm{l} \end{pmatrix} \qquad p_\mathrm{v}^\mu = \begin{pmatrix} E_\mathrm{v} \\ E_\mathrm{v} \sin\theta \\ 0 \\ E_\mathrm{v} \cos\theta \end{pmatrix} \qquad p_\mathrm{c}^\mu = p_\mathrm{b}^\mu - p_\mathrm{l}^\mu - p_\mathrm{v}^\mu \tag{4.41}$$

Interchanging the parametrization of the leptons makes no difference due to the assumption of massless leptons. But swapping the parametrization of the c-quark momentum p_c with one of the lepton momenta increases the complexity of the expression to integrate sizeably because of the c-quark mass. The expressions would be

$$p_\mathrm{c}^\mu = \begin{pmatrix} E_c \\ 0 \\ 0 \\ \sqrt{E_c^2 - m_\mathrm{c}^2} \end{pmatrix}, \qquad \text{or even worse} \qquad p_\mathrm{c}^\mu = \begin{pmatrix} E_c \\ \sqrt{E_c^2 - m_\mathrm{c}^2}\sin\theta \\ 0 \\ \sqrt{E_c^2 - m_\mathrm{c}^2}\cos\theta \end{pmatrix}. \tag{4.42}$$

These parametrizations would slow down the Monte-Carlo based phase space integration. This is especially reasonable for the calculation of the real corrections causing a slowdown of a factor of around ten. We will keep all parametrizations throughout this work similar, hence we discuss the optimization of the structure here.

The phase space integral is

$$\begin{aligned} \mathrm{d}\Pi(3) &= \widetilde{\mathrm{d}p_\mathrm{l}}\,\widetilde{\mathrm{d}p_\mathrm{v}}\,\widetilde{\mathrm{d}p_{X_\mathrm{c}}}\,(4\pi)^4\delta^4(p_\mathrm{b} - p_\mathrm{l} - p_\mathrm{v} - p_{X_\mathrm{c}}) \\ &= \frac{\mathrm{d}^4 p_\mathrm{l}}{(2\pi)^3}\delta(p_\mathrm{l}^2)\Theta(p_{\mathrm{l},0})\,\frac{\mathrm{d}^4 p_\mathrm{v}}{(2\pi)^3}\delta(p_\mathrm{v}^2)\Theta(p_{\mathrm{v},0})\,\frac{\mathrm{d}^4 p_{X_\mathrm{c}}}{(2\pi)^3}\Theta(p_{X_\mathrm{c},0})\,(2\pi)^4\delta^4(p_\mathrm{b}-p_\mathrm{l}-p_\mathrm{v}-p_{X_\mathrm{c}}) \end{aligned}$$

where we do not assume the on-shellness of the hadronic final state containing a charm quark X_c, because this is taken care of by the delta functions arising from the imaginary part of the background field propagator S_BGF. The derivatives of the delta function have to be transferred to the integrand by partial integration, which has to be done prior to putting the hadronic final state on its mass shell. In the case of the tree level expansion presented in this section the final hadronic state is simply the charm quark itself.

integrating over $p_{l,0}$, $p_{\nu,0}$ and using the four-momentum conservation to integrate over p_{X_c}:

$$\begin{aligned} d\Pi(3) &= \frac{d^3 p_l}{(2\pi)^3 2E_l} \frac{d^3 p_\nu}{(2\pi)^3 2E_\nu}\Bigg|_{p_{X_c}=p_b-p_l-p_\nu} \\ &= \frac{d\Omega_l \, |\boldsymbol{p}_l|^2 \, d|\boldsymbol{p}_l|}{(2\pi)^3 2E_l} \frac{d\Omega_\nu \, |\boldsymbol{p}_\nu|^2 \, d|\boldsymbol{p}_\nu|}{(2\pi)^3 2E_\nu}\Bigg|_{p_{X_c}=p_b-p_l-p_\nu} \\ &= 8\pi^2 \frac{E_l \, dE_l}{2(2\pi)^3} \frac{E_\nu \, dE_\nu \, d\cos\theta}{2(2\pi)^3}\Bigg|_{p_{X_c}=p_b-p_l-p_\nu} \end{aligned} \tag{4.43}$$

The range of the lepton energy E_l in this decay is[2]

$$0 \le E_l \le \frac{m_b}{2}(1-\rho) \tag{4.44}$$

with $\rho = m_c^2/m_b^2$. The range of the antineutrino energy is determined by the charged lepton energy E_l, the angle θ and $m^2_{X_c,\min} = m_c^2$, the minimum invariant mass of the hadronic final state[3]:

$$0 \le E_\nu \le \frac{\frac{m_b}{2}(1-\rho) - E_l}{1 - \frac{E_l}{m_b}(1-\cos\theta)}. \tag{4.45}$$

The range of the third integration variable $\cos\theta$ is

$$-1 \le \cos\theta \le 1. \tag{4.46}$$

The on-shell relation for the charm quark as appearing in the delta function from the imaginary part of the propagator expansion becomes

$$p_c^2 - m_c^2 = (p_b - p_l - p_\nu)^2 - m_c^2 = m_b^2(1-\rho) - 2m_b E_l - 2m_b \kappa_1 E_\nu$$

with $\kappa_1 = 1 - \frac{E_l}{m_b}(1-\cos\theta)$. Thus the delta function itself is:

$$\mathrm{Im}\left(\frac{1}{p_c^2 - m_c^2 + i\epsilon}\right)^{(n+1)} = -\frac{\pi}{|2m_b\kappa_1|^{n+1}} \frac{(-1)^n}{n!} \delta^{(n)}\left(E_\nu - \frac{m_b}{2\kappa_1}\left((1-\rho) - \frac{2}{m_b}E_l\right)\right). \tag{4.47}$$

For the Monte-Carlo integration of the phase space it is convenient to substitute the integration variables by dimensionless and normalized ones. Especially integration limits depending on subsequent integration variables cannot be handled by the multi-dimensional integration routines we use [28]. This can be done in the manner of

$$\int_a^b f(x)dx \longrightarrow \int_0^1 (b-a) f(z) dz \qquad \text{with} \qquad z = \frac{x-a}{b-a}. \tag{4.48}$$

For this purpose a procedure introduces integration variables running between 0 and 1 and takes care of the right substitutions.

[2] The upper limit can be derived from squaring the four-momentum conservation $(p_b - p_l)^2 = (p_\nu + p_{X_c})^2$ with $E_\nu = 0$ and $m_{X_c}^2 = m_c^2$.

[3] Analogously to E_l the upper limit of E_ν can be derived from $(p_b - p_\nu)^2 = (p_l + p_{X_c})^2$ with E_l from (4.44) and $m_{X_c}^2 = m_c^2$.

4.5 Results

4.5.1 Total rate

The total decay rate is

$$\Gamma = \int \mathrm{d}E_{\mathrm{l}} \int \mathrm{d}E_{\nu} \int \mathrm{d}\cos\Theta \frac{\mathrm{d}\Gamma}{\mathrm{d}E_{\mathrm{l}}\mathrm{d}E_{\nu}\,\mathrm{d}\cos\Theta}. \tag{4.49}$$

Due to squaring the amplitude in the hadronic tensor (4.3) the parameters of the enhanced current c_{L}, c_{R}, g_{L}, g_{R}, d_{L} and d_{R} appear in pairs. The term proportional to c_{L}^2 yields the standard model result. Assuming the correction to the left-handed current to be small, we only include the interference terms with the standard model term, i. e. $c_{\mathrm{L}}c_{\mathrm{R}}$, $c_{\mathrm{L}}g_{\mathrm{L}}$,$c_{\mathrm{L}}g_{\mathrm{R}}$, $c_{\mathrm{L}}d_{\mathrm{L}}$ and $c_{\mathrm{L}}d_{\mathrm{R}}$. As we have argued in the effective field theory derivation of the enhanced current the new contributions are of order $1/\Lambda^2$. The square of these new-physics terms is then already of order $1/\Lambda^4$ which can be neglected in favor of the interference term with the standard model term, which is of order $1/\Lambda^2$.

The total rate can be decomposed into contributions from these parameter combinations:

$$\Gamma = \frac{G_{\mathrm{F}}m_{\mathrm{b}}^5|V_{cb}|}{192\pi^3}\left(c_{\mathrm{L}}^2\Gamma^{c_{\mathrm{L}}c_{\mathrm{L}}} + c_{\mathrm{L}}c_{\mathrm{R}}\Gamma^{c_{\mathrm{L}}c_{\mathrm{R}}} + c_{\mathrm{L}}g_{\mathrm{L}}\Gamma^{c_{\mathrm{L}}g_{\mathrm{L}}} + c_{\mathrm{L}}g_{\mathrm{R}}\Gamma^{c_{\mathrm{L}}g_{\mathrm{R}}} + c_{\mathrm{L}}d_{\mathrm{L}}\Gamma^{c_{\mathrm{L}}d_{\mathrm{L}}} + c_{\mathrm{L}}d_{\mathrm{R}}\Gamma^{c_{\mathrm{L}}d_{\mathrm{R}}}\right) \tag{4.50}$$

which are

$$\begin{aligned}
\Gamma^{c_{\mathrm{L}}c_{\mathrm{L}}} =&\quad -\rho^4 + 8\rho^3 - 12\rho^2\log(\rho) - 8\rho + 1\\
&-(-\rho^4 + 8\rho^3 - 12\rho^2\log(\rho) - 8\rho + 1)\frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2}\\
&+(\ -5\rho^4 + 24\rho^3 - 24\rho^2 - 12\rho^2\log(\rho) + 8\rho - 3)\frac{\hat{\mu}_{\mathrm{G}}^2}{3m_{\mathrm{b}}^2}\\
\Gamma^{c_{\mathrm{L}}c_{\mathrm{R}}} =&\quad 4\sqrt{\rho}\left(\rho^3 + 9\rho^2 - 9\rho - 6(\rho+1)\rho\log(\rho) - 1\right)\\
&-4\sqrt{\rho}\left(\rho^3 + 9\rho^2 - 9\rho - 6(\rho+1)\rho\log(\rho) - 1\right)\frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2}\\
&+\frac{4}{3}\sqrt{\rho}\left(13\rho^3 - 27\rho^2 - 6\left(3\rho^2 - 3\rho + 2\right)\log(\rho) + 27\rho - 13\right)\frac{\hat{\mu}_{\mathrm{G}}^2}{3m_{\mathrm{b}}^2}\\
\Gamma^{c_{\mathrm{L}}g_{\mathrm{L}}} =&\ \left(-\rho^4 + 8\rho^3 - 12\rho^2\log(\rho) - 8\rho + 1\right)\\
&-\frac{4}{3}\left(\rho^4 - 6\rho^3 + 18\rho^2 - 10\rho - 12\rho\log(\rho) - 3\right)\frac{\hat{\mu}_{\mathrm{G}}^2}{2m_{\mathrm{b}}^2}\\
\Gamma^{c_{\mathrm{L}}g_{\mathrm{R}}} =&\ \sqrt{\rho}\left(-\rho^4 + 8\rho^3 - 12\rho^2\log(\rho) - 8\rho + 1\right)\\
&-\frac{20}{3}\sqrt{\rho}\left(\rho^4 - 6\rho^3 + 18\rho^2 - 10\rho - 12\rho\log(\rho) - 3\right)\frac{\hat{\mu}_{\mathrm{G}}^2}{2m_{\mathrm{b}}^2}\\
\Gamma^{c_{\mathrm{L}}d_{\mathrm{L}}} =&\ +\left(3\rho^4 + 44\rho^3 - 24\rho^3\log(\rho) - 36\rho^2 - 36\rho^2\log(\rho) - 12\rho + 1\right)\\
&-\left(3\rho^4 + 44\rho^3 - 24\rho^3\log(\rho) - 36\rho^2 - 36\rho^2\log(\rho) - 12\rho + 1\right)\frac{\hat{\mu}_\pi^2}{2m_{\mathrm{b}}^2}\\
&+\frac{1}{3}\left(29\rho^4 - 12\rho^3 - 72\rho^3\log(\rho) + 36\rho^2 + 36\rho^2\log(\rho) - 20\rho - 96\rho\log(\rho) - 33\right)\frac{\hat{\mu}_{\mathrm{G}}^2}{2m_{\mathrm{b}}^2}\\
\Gamma^{c_{\mathrm{L}}d_{\mathrm{R}}} =&\ +\sqrt{\rho}\left(-\rho^4 + 12\rho^3 + 36\rho^2 - 36\rho^2\log(\rho) - 44\rho - 24\rho\log(\rho) - 3\right)\\
&-\sqrt{\rho}\left(-\rho^4 - 12\rho^3 + 36\rho^2 - 36\rho^2\log(\rho) - 44\rho - 24\rho\log(\rho) - 3\right)\frac{\hat{\mu}_\pi^2}{2m_{\mathrm{b}}^2}\\
&+\frac{1}{3}\sqrt{\rho}\left(-15\rho^4 + 100\rho^3 - 36\rho^2 - (108\rho^2 + 72\rho + 48)\log(\rho) + 60\rho - 109\right)\frac{\hat{\mu}_{\mathrm{G}}^2}{2m_{\mathrm{b}}^2}
\end{aligned} \tag{4.51}$$

As discussed in the effective field theory derivation of the enhanced current, the interference terms from right-handed currents are suppressed by $m_c/m_b = \sqrt{\rho}$ accounting for the needed helicity flip, which we can see in the above displayed parts of the total rate.

4.5.2 Lepton Energy Spectrum

It is also instructive to look at the lepton energy spectrum, as the moments of this spectrum are measured with high precision. Especially a plot of the various contributions show how they would effect the measured moments. The lepton energy spectrum can as well as the total rate be decomposed into the contributions of the parameter combinations with $y = \frac{2}{m_b}E_l$:

$$\begin{aligned}\frac{d\Gamma}{dy} &= \frac{G_F m_b^5 |V_{cb}|}{192\pi^3}\left(c_L^2 \frac{d\Gamma^{c_L c_L}}{dy} + c_L c_R \frac{d\Gamma^{c_L c_R}}{dy} + c_L g_L \frac{d\Gamma^{c_L g_L}}{dy}\right. \\ &\qquad \left. + c_L g_R \frac{d\Gamma^{c_L g_R}}{dy} + c_L d_L \frac{d\Gamma^{c_L d_L}}{dy} + c_L d_R \frac{d\Gamma^{c_L d_R}}{dy}\right)\end{aligned} \tag{4.52}$$

$$\begin{aligned}\frac{d\Gamma^{c_L c_L}}{dy} &= \frac{2(y-3)y^2\rho^3}{(y-1)^3} - \frac{6y^2\rho^2}{(y-1)^2} - 6y^2\rho - 2y^2(2y-3) \\ &+ \left(\frac{2(2y-5)y^3\rho^2}{(y-1)^4} - \frac{10y^3}{3} - \frac{4\left(y^2-5y+10\right)y^3\rho^3}{3(y-1)^5}\right)\frac{\hat{\mu}_\pi^2}{m_b^2} \\ &+ \left(\frac{10y^2\left(y^2-4y+6\right)\rho^3}{(y-1)^4} - \frac{18(y-2)y^2\rho^2}{(y-1)^3} + \frac{12y^2(2y-3)\rho}{(y-1)^2} + 2y^2(5y+6)\right)\frac{\hat{\mu}_G^2}{3m_b^2} \\ \frac{d\Gamma^{c_L c_R}}{dy} &= \sqrt{\rho}\left(-\frac{12y^2\rho^2}{(y-1)^2} - \frac{24y^2\rho}{y-1} - 12y^2\right) \\ &+ \sqrt{\rho}\left(\frac{4(2y-5)y^3\rho^2}{(y-1)^4} + \frac{4(3y-5)y^3\rho}{(y-1)^3}\right)\frac{\hat{\mu}_\pi^2}{m_b^2} \\ &+ \sqrt{\rho}\left(\frac{12y^3\rho}{(y-1)^2} - \frac{36(y-2)y^2\rho^2}{(y-1)^3} + \frac{24(2y-3)y^2}{y-1}\right)\frac{\hat{\mu}_G^2}{3m_b^2} \\ \frac{d\Gamma^{c_L g_L}}{dy} &= \left(-\frac{12y^2\rho^2}{y-1} - 24y^2\rho - 12(y-1)y^2\right) \\ &+ \left(\frac{2\left(4y^2-9y+3\right)y^2\rho^2}{(y-1)^3} + 12y^2\rho - 6y^2\right)\frac{\hat{\mu}_\pi^2}{m_b^2} \\ &+ \left(-\frac{6(2y-3)y^2\rho^2}{(y-1)^2} - \frac{12(y-3)y^2\rho}{y-1} + 18y^2\right)\frac{\hat{\mu}_G^2}{3m_b^2} \\ \frac{d\Gamma^{c_L g_R}}{dy} &= \sqrt{\rho}\left(-\frac{12y^2\rho^2}{y-1} - 24y^2\rho - 12(y-1)y^2\right) \\ &+ \sqrt{\rho}\left(\frac{2\left(4y^2-9y+3\right)y^2\rho^2}{(y-1)^3} + 12y^2\rho - 6y^2\right)\frac{\hat{\mu}_\pi^2}{m_b^2} \\ &+ \sqrt{\rho}\left(-\frac{30(2y-3)y^2\rho^2}{(y-1)^2} - \frac{60(y-3)y^2\rho}{y-1} + 90y^2\right)\frac{\hat{\mu}_G^2}{3m_b^2} \\ \frac{d\Gamma^{c_L d_L}}{dy} &= \left(-\frac{8y^3\rho^3}{(y-1)^3} - \frac{12y^3\rho^2}{(y-1)^2} + 4y^3\right) \\ &+ \left(\frac{10y^3}{3} + \frac{4\left(4y^2-11y-5\right)y^3\rho^3}{3(y-1)^5} + \frac{2\left(3y^2-4y-5\right)y^3\rho^2}{(y-1)^4}\right)\frac{\hat{\mu}_\pi^2}{m_b^2} \\ &+ \left(-\frac{12(2y-5)y^3\rho^3}{(y-1)^4} - \frac{6(y-3)y^3\rho^2}{(y-1)^3} + \frac{24(y-2)y^3\rho}{(y-1)^2} + 6y^3\right)\frac{\hat{\mu}_G^2}{3m_b^2}\end{aligned}$$

$$
\begin{aligned}
\frac{\mathrm{d}\Gamma^{c_\mathrm{L}d_\mathrm{R}}}{\mathrm{d}y} = &\sqrt{\rho}\left(\frac{4(y-3)y^2\rho^3}{(y-1)^3}+\frac{12(y-3)y^2\rho^2}{(y-1)^2}+\frac{12(y-3)y^2\rho}{y-1}+4(y-3)y^2\right)\\
&+\sqrt{\rho}\left(\frac{4(3y-5)y^3\rho}{(y-1)^3}+\frac{10y^3}{3}-\frac{8\left(y^2-5y+10\right)y^3\rho^3}{3(y-1)^5}\right.\\
&\left.-\frac{2\left(3y^2-16y+25\right)y^3\rho^2}{(y-1)^4}\right)\frac{\hat{\mu}_\pi^2}{m_\mathrm{b}^2}\\
&+\sqrt{\rho}\left(\frac{20y^2\left(y^2-4y+6\right)\rho^3}{(y-1)^4}+\frac{6y^2\left(5y^2-25y+36\right)\rho^2}{(y-1)^3}\right.\\
&\left.-\frac{12y^2(5y-6)\rho}{(y-1)^2}-\frac{2y^2\left(5y^2-5y+12\right)}{y-1}\right)\frac{\hat{\mu}_G^2}{3m_\mathrm{b}^2}
\end{aligned}
$$

4.5.3 Cut on the Lepton Energy

The experimental determination of the leptonic energy spectrum and thus its moments is difficult for leptons with low energy. The leptons with low energy drown in the background from other processes in the detector. The standard circumvention is to measure only the leptons with a certain minimal energy. The theoretical description has to account for this procedure. Therefore we impose a lower energy cut $E_\mathrm{cut}=\frac{m_\mathrm{b}}{2}\xi$ on the charged lepton energy:

$$
\frac{m_\mathrm{b}}{2}\xi \le E_\mathrm{l} \le \frac{m_\mathrm{b}}{2}(1-\rho). \tag{4.53}
$$

We present here only the standard-model total rate with such a cut as an example:

$$
\begin{aligned}
\Gamma^{c_\mathrm{L}c_\mathrm{L}}_{E_\mathrm{cut}} = &+\frac{1}{(\xi-1)^2}\big(\xi^6+2\xi^5\rho-4\xi^5-4\xi^4\rho+5\xi^4-2\xi^3\rho^3+6\xi^3\rho^2+2\xi^3\rho-2\xi^3-\xi^2\rho^4+8\xi^2\rho^3\\
&-18\xi^2\rho^2+12\xi^2\rho^2\log\left(\frac{1-\xi}{\rho}\right)-8\xi^2\rho+\xi^2+2\xi\rho^4-16\xi\rho^3+12\xi\rho^2\\
&-24\xi\rho^2\log\left(\frac{1-\xi}{\rho}\right)+12\rho^2\log\left(\frac{1-\xi}{\rho}\right)+16\xi\rho-2\xi-\rho^4+8\rho^3-8\rho+1\big)\\
&+\frac{1}{6(\xi-1)^4}\frac{\mu_\pi^2}{2\mathrm{mb}^2}\big(5\xi^8-20\xi^7+30\xi^6+8\xi^5\rho^3-24\xi^5\rho^2-20\xi^5+3\xi^4\rho^4-44\xi^4\rho^3\\
&+90\xi^4\rho^2-36\xi^4\rho^2\log\left(\frac{1-\xi}{\rho}\right)+24\xi^4\rho+2\xi^4-12\xi^3\rho^4+96\xi^3\rho^3\\
&-156\xi^3\rho^2+144\xi^3\rho^2\log\left(\frac{1-\xi}{\rho}\right)-96\xi^3\rho+12\xi^3+18\xi^2\rho^4\\
&-144\xi^2\rho^3+126\xi^2\rho^2-216\xi^2\rho^2\log\left(\frac{1-\xi}{\rho}\right)+144\xi^2\rho-18\xi^2\\
&-12\xi\rho^4+96\xi\rho^3-36\xi\rho^2+144\xi\rho^2\log\left(\frac{1-\xi}{\rho}\right)-36\rho^2\log\left(\frac{1-\xi}{\rho}\right)\\
&-96\xi\rho+12\xi+3\rho^4-24\rho^3+24\rho-3\big)\\
&-\frac{1}{6(\xi-1)^3}\frac{\hat{\mu}_G^2}{2\mathrm{mb}^2}\big(5\xi^7-7\xi^6+24\xi^5\rho-9\xi^5+20\xi^4\rho^3-36\xi^4\rho^2-48\xi^4\rho+19\xi^4+15\xi^3\rho^4\\
&-112\xi^3\rho^3+162\xi^3\rho^2-36\xi^3\rho^2\log\left(\frac{1-\xi}{\rho}\right)+\xi^3-45\xi^2\rho^4+216\xi^2\rho^3\\
&-306\xi^2\rho^2+108\xi^2\rho^2\log\left(\frac{1-\xi}{\rho}\right)+72\xi^2\rho-27\xi^2+45\xi\rho^4-216\xi\rho^3\\
&+252\xi\rho^2-108\xi\rho^2\log\left(\frac{1-\xi}{\rho}\right)+36\rho^2\log\left(\frac{1-\xi}{\rho}\right)-72\xi\rho+27\xi\\
&-15\rho^4+72\rho^3-72\rho^2+24\rho-9\big)
\end{aligned}
\tag{4.54}
$$

4.5.4 Moments

For the combined fit in the final analysis the central moments of the charged lepton energy and the central moments of the hadronic invariant mass are needed up to a power of three. These are for the *charged lepton energy moments*:

order	moment
0	Γ
1	$\langle E_l \rangle$
2	$\langle (E_l - \langle E_l \rangle)^2 \rangle$
3	$\langle (E_l - \langle E_l \rangle)^3 \rangle$

(4.55)

and the *moments of the hadronic invariant mass*:

order	moment
0	Γ
1	$\langle m_X^2 \rangle$
2	$\left\langle \left(m_X^2 - \langle m_X^2 \rangle\right)^2 \right\rangle$
3	$\left\langle \left(m_X^2 - \langle m_X^2 \rangle\right)^3 \right\rangle$

(4.56)

For the evaluation of the moments of the charged lepton energy it is sufficient to compute the non-central values, because they are related to each other by expansion of the weight as follows:

$$\langle (E_l - \langle E_l \rangle)^2 \rangle = \langle E_l^2 \rangle - \langle E_l \rangle^2 \tag{4.57}$$

$$\langle (E_l - \langle E_l \rangle)^3 \rangle = \langle E_l^3 \rangle - 3\langle E_l^2 \rangle \langle E_l \rangle + 2\langle E_l \rangle^3. \tag{4.58}$$

In order to explain the relation between the calculated moments and the moments needed in the analysis for the hadronic invariant mass we have to discuss the connection between hadronic and partonic variables. The hadronic energy and the hadronic invariant mass of the decay products can be written as

$$\begin{aligned} E_{Had} &= v \cdot (p_B - q) = m_B - v \cdot q \\ m_X^2 = s_{Had} &= (p_B - q)^2 = m_B^2 - 2m_B\, v \cdot q + q^2, \end{aligned} \tag{4.59}$$

where m_B and $p_B = m_B v$ are the mass and the momentum of the B meson and q is the momentum of the leptonic system. The corresponding variables on parton level in a dimensionless form are

$$\begin{aligned} \hat{E}_0 &= \frac{E_0}{m_b} = \frac{v \cdot (p_b - q)}{m_b} = 1 - v \cdot \hat{q} \\ \hat{s}_0 &= \frac{s_0}{m_b^2} = \frac{(p_b - q)^2}{m_b^2} = 1 - 2v \cdot \hat{q} + \hat{q}^2, \end{aligned} \tag{4.60}$$

where p_b is the the b-quark momentum. To relate the hadronic variables to the partonic ones (4.60) can be solved as $v \cdot q = m_b(1 - \hat{E}_0)$ and $q^2 = m_b^2(s - 2\hat{E}_0 + 1)$ and substituted into (4.59). The B-Meson mass can be expanded as

$$m_B = m_b + \Lambda + \ldots, \tag{4.61}$$

where m_b is the b-quark mass. Using this for (4.59) it is possible to relate the hadronic variables in (4.59) to the partonic ones:

$$\begin{aligned} E_{Had} &= \Lambda + m_{\rm b}\,\hat{\rm E}_0 \\ m_X^2 = s_{Had} &= m_{\rm c}^2 + \Lambda^2 + 2 m_{\rm b} \Lambda\,\hat{\rm E}_0 + m_{\rm b}^2 (\hat{\rm s}_0 - \rho). \end{aligned} \tag{4.62}$$

In the last term for m_X^2 we inserted a 1 in form of $m_{\rm b}^2\rho - m_{\rm b}^2\rho$ to express it in terms of the centered weight $(\hat{\rm s}_0 - \rho)$. Thus the moments of the hadronic invariant mass can be related to moments of the partonic energy and partonic invariant mass:

$$\langle m_X^2 \rangle = m_c^2 + \Lambda^2 + 2\Lambda m_b \langle E_0 \rangle + m_b^2 \langle \hat{\rm s}_0 - \rho \rangle \tag{4.63}$$

$$\begin{aligned} \left\langle \left(m_X^2 - \langle m_X^2\rangle\right)^2 \right\rangle = \; & m_b^4 \left(\langle (\hat{\rm s}_0 - \rho)^2 \rangle - \langle \hat{\rm s}_0 - \rho \rangle^2 \right) \\ & + 4\Lambda m_b^3 \left(\langle \hat{E}_0 (\hat{\rm s}_0 - \rho) \rangle - \langle \hat{E}_0 \rangle \langle \hat{\rm s}_0 - \rho \rangle \right) \\ & + 4\Lambda^2 m_b^2 \left(\langle \hat{E}_0^2 \rangle - \langle \hat{E}_0 \rangle^2 \right) \end{aligned} \tag{4.64}$$

$$\begin{aligned} \left\langle \left(m_X^2 - \langle m_X^2\rangle\right)^3 \right\rangle = \; & 12\Lambda^2 m_b^4 \left(2\langle \hat{E}_0\rangle^2 \langle \hat{\rm s}_0 - \rho\rangle - 2\langle \hat{E}_0 \rangle \langle \hat{E}_0 (\hat{\rm s}_0 - \rho)\rangle - \langle \hat{E}_0^2 \rangle \langle \hat{\rm s}_0 - \rho \rangle + \langle \hat{E}_0^2\, \hat{\rm s}_0 - \rho \rangle \right) \\ & + 6\Lambda m_b^5 \Big(- 2 \langle \hat{\rm s}_0 - \rho \rangle \langle \hat{E}_0 (\hat{\rm s}_0 - \rho) \rangle \\ & \qquad + \langle \hat{E}_0 \rangle \left(2\langle \hat{\rm s}_0 - \rho \rangle^2 - \langle (\hat{\rm s}_0 - \rho)^2 \rangle \right) + \langle \hat{E}_0 (\hat{\rm s}_0 - \rho)^2 \rangle \Big) \\ & + 8\Lambda^3 m_b^3 \left(2\langle \hat{E}_0 \rangle^3 - 3 \langle \hat{E}_0^2 \rangle \langle \hat{E}_0 \rangle + \langle \hat{E}_0^3 \rangle \right) \\ & + m_b^6 \left(2\langle \hat{\rm s}_0 - \rho \rangle^3 - 3 \langle (\hat{\rm s}_0 - \rho)^2 \rangle \langle \hat{\rm s}_0 - \rho \rangle + \langle (\hat{\rm s}_0 - \rho)^3 \rangle \right) . \end{aligned} \tag{4.65}$$

The partonic moments on the right needed for the calculations of the hadronic moments are called *building blocks*, which we will compute and display in the following, were we shall quote the results in terms of the partonic variables (4.60).

In table 5.1 we list the numerical results for the moments of the lepton energy spectrum

$$L_n = \frac{\langle \hat{E}_{\rm l}^n \rangle}{\Gamma_0} = \frac{1}{\Gamma_0} \int_{E_{\rm cut}} {\rm d}E_{\rm l} \int {\rm d}E_\nu \int {\rm d}\cos\Theta\; \hat{E}_{\rm l}^n \,\frac{{\rm d}\Gamma}{{\rm d}E_{\rm l}{\rm d}E_\nu\,{\rm d}\cos\Theta} \tag{4.66}$$

with the dimensionless lepton energy $\hat{E}_{\rm l} = E_{\rm l}/m_{\rm b}$ and in table 4.2 we quote the *building blocks* of the moments of the hadronic invariant mass:

$$H_{ij} = \frac{\langle (\hat{\rm s}_0 - \rho)^i \hat{E}_0^j \rangle}{\Gamma_0} = \frac{1}{\Gamma_0} \int_{E_{\rm cut}} {\rm d}E_{\rm l} \int {\rm d}E_\nu \int {\rm d}\cos\Theta\; (\hat{s}_0 - \rho)^i\, \hat{E}_0^j \,\frac{{\rm d}\Gamma}{{\rm d}E_{\rm l}{\rm d}E_\nu\,{\rm d}\cos\Theta} \tag{4.67}$$

where the normalization

$$\Gamma_0 = \frac{G_{\rm F}^2 |V_{cb}|^2 m_b^5}{192\pi^3} \left(1 - 8\rho - 12\rho^2 \ln\rho + 8\rho^3 - \rho^4\right) \tag{4.68}$$

is given in terms of the partonic rate.

For the fit with experimental data we need analytical expressions of the moments depending on ρ and the cut on the charged lepton energy ξ ($E_{\rm cut} = \frac{2}{m_{\rm b}}\xi$), which are not displayed here due to their bulkiness. As an example we refer to the analytical expression of the standard-model total rate with cut in (4.54).

		n	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
no E_l cuts	parton	0	1.00000	−0.66845	1.00000	0.25000	0.83289	−0.41845
		1	0.30720	−0.20923	0.27079	0.06770	0.29131	−0.12332
		2	0.10300	−0.07076	0.08167	0.02042	0.10664	−0.03968
		3	0.03652	−0.02517	0.02645	0.00661	0.04031	−0.01352
	μ_π^2/m_b^2	0	−0.50000	0.33423	0.00000	0.00000	−0.41644	0.20923
		1	0.00000	0.00000	0.22566	0.05641	0.00000	0.00000
		2	0.08583	−0.05897	0.16335	0.04084	0.08886	−0.03307
		3	0.07305	−0.05035	0.09256	0.02314	0.08061	−0.02705
	μ_g^2/m_b^2	0	−1.94492	4.99340	1.55793	1.94742	−4.69943	2.72749
		1	−0.96251	1.85784	0.50462	0.63078	−2.01032	1.01144
		2	−0.44952	0.72367	0.17605	0.22006	−0.86632	0.39025
		3	−0.20521	0.29019	0.06442	0.08052	−0.37562	0.15453
$E_l > 1$ GeV cut	parton	0	0.81481	−0.56169	0.71136	0.17784	0.77784	−0.33213
		1	0.27764	−0.19193	0.22556	0.05639	0.28175	−0.10950
		2	0.09793	−0.06777	0.07402	0.01850	0.10491	−0.03731
		3	0.03562	−0.02463	0.02509	0.00627	0.03998	−0.01310
	μ_π^2/m_b^2	0	−0.45038	0.32245	0.22803	0.05701	−0.45987	0.18733
		1	0.00866	−0.00214	0.26282	0.06570	−0.00751	−0.00388
		2	0.08741	−0.05937	0.16981	0.04245	0.08751	−0.03378
		3	0.07334	−0.05042	0.09373	0.02343	0.08036	−0.02718
	μ_g^2/m_b^2	0	−2.10292	4.69026	1.33948	1.67435	−4.69883	2.59001
		1	−0.98834	1.80778	0.46915	0.58644	−2.01006	0.98846
		2	−0.45402	0.71489	0.16987	0.21234	−0.86625	0.38618
		3	−0.20602	0.28859	0.06331	0.07913	−0.37561	0.15377

Table 4.1: *Tree level coefficients of the leptonic moments without E_l cuts and with a cut $E_l > 1\,\mathrm{GeV}$.*

The entries in the tables contain the coefficients corresponding to the expansion of the various moments:

$$L_n = c_L^2 L_n^{c_L c_L} + c_L c_R L_n^{c_L c_R} + c_L d_L L_n^{c_L d_L} + c_L d_R L_n^{c_L d_R} + c_L g_L L_n^{c_L g_L} + c_L g_R L_n^{c_L g_R} \tag{4.69}$$

$$H_{ij} = c_L^2 H_{ij}^{c_L c_L} + c_L c_R H_{ij}^{c_L c_R} + c_L d_L H_{ij}^{c_L d_L} + c_L d_R H_{ij}^{c_L d_R} + c_L g_L H_{ij}^{c_L g_L} + c_L g_R H_{ij}^{c_L g_R} \tag{4.70}$$

where all the coefficients have an expansion in $1/m_b$

$$\begin{aligned} L_n^{(c_1 c_2)} &= L_n^{(c_1 c_2;\, m_\mathrm{b}^0)} + \frac{\hat{\mu}_\pi^2}{m_\mathrm{b}^2} L_n^{(c_1 c_2;\, m_\mathrm{b}^2)} + \frac{\hat{\mu}_\mathrm{G}^2}{m_\mathrm{b}^2} L_n^{(c_1 c_2;\, m_\mathrm{b}^2)} + \ldots \\ H_{ij}^{(c_1 c_2)} &= H_{ij}^{(c_1 c_2;\, m_\mathrm{b}^0)} + \frac{\hat{\mu}_\pi^2}{m_\mathrm{b}^2} H_{ij}^{(c_1 c_2;\, m_\mathrm{b}^2)} + \frac{\hat{\mu}_\mathrm{G}^2}{m_b^2} H_{ij}^{(c_1 c_2;\, m_\mathrm{b}^2)} + \ldots \end{aligned}$$

Please note that the moments computed for the non-vector couplings at this stage are not very significant, because we have to include the radiative corrections, which show that these coupling operators mix by changing the renormalization scale. Thus the moments displayed here are the values at the high renormalization scale Λ.

		i	j	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
no E_l cuts	parton	0	0	1.00000	−0.66845	1.00000	0.25000	0.83289	−0.41845
		0	1	0.42201	−0.25000	0.45842	0.11461	0.32310	−0.15360
		0	2	0.18319	−0.09640	0.21336	0.05334	0.12892	−0.05814
		0	3	0.08147	−0.03825	0.10060	0.02515	0.05279	−0.02267
		$i>0$	j	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	μ_π^2/m_b^2 coeff.	0	0	−0.50000	0.33423	0.00000	0.00000	−0.41644	0.20923
		0	1	−0.50000	0.33423	−0.45131	−0.11283	−0.41644	0.20922
		0	2	−0.29018	0.18359	−0.35814	−0.08954	−0.23301	0.11387
		0	3	−0.13821	0.08371	−0.20996	−0.05248	−0.10800	0.05148
		1	0	−0.57799	0.41845	−0.90263	−0.22566	−0.50979	0.26485
		1	1	−0.15836	0.11719	−0.30785	−0.07702	−0.14293	0.07413
		1	2	−0.02825	0.02797	−0.09201	−0.02300	−0.03268	0.01805
		2	0	0.16092	−0.07283	0.20115	0.05029	0.10248	−0.04265
		2	1	0.07346	−0.03018	0.09593	0.02404	0.04345	−0.01741
		3	0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	μ_g^2/m_b^2 coeff.	0	0	−1.94492	4.99340	1.55793	1.94742	−4.69943	2.72749
		0	1	−0.38504	1.27772	0.54869	0.68586	−1.22854	0.66151
		0	2	−0.03022	0.28327	0.18909	0.23636	−0.28540	0.14051
		0	3	0.02976	0.03423	0.06169	0.07711	−0.04391	0.01846
		1	0	0.31431	−0.63948	−0.18053	−0.22566	0.55217	−0.25667
		1	1	0.11954	−0.25614	−0.08170	−0.10213	0.21650	−0.09426
		1	2	0.04656	−0.10595	−0.03760	−0.04698	0.08726	−0.03566
		$i>1$	j	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
$E_l > 1\,\mathrm{GeV}$ cut	parton	0	0	0.81481	−0.56169	0.71136	0.17784	0.77784	−0.33213
		0	1	0.33408	−0.20370	0.31872	0.07968	0.29852	−0.11634
		0	2	0.14111	−0.07612	0.14535	0.03634	0.11785	−0.04190
		0	3	0.06120	−0.02929	0.06731	0.01683	0.04776	−0.01552
		$i>0$	j	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	μ_π^2/m_b^2 coeff.	0	0	−0.45038	0.32245	0.22803	0.05701	−0.45987	0.18733
		0	1	−0.45054	0.29207	−0.26905	−0.06726	−0.42195	0.16833
		0	2	−0.26732	0.15609	−0.25415	−0.06354	−0.23320	0.08803
		0	3	−0.13371	0.07060	−0.16106	−0.04027	−0.10876	0.03900
		1	0	−0.54242	0.35896	−0.76454	−0.19114	−0.49232	0.21344
		1	1	−0.16385	0.10217	−0.27657	−0.06914	−0.14116	0.06049
		1	2	−0.04171	0.02620	−0.09426	−0.02357	−0.03463	0.01590
		2	0	0.12025	−0.05469	0.13452	0.03363	0.09231	−0.02819
		2	1	0.05375	−0.02207	0.06319	0.01580	0.03880	−0.01099
		3	0	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000
	μ_g^2/m_b^2 coeff.	0	0	−2.10292	4.69026	1.33948	1.67435	−4.69883	2.59001
		0	1	−0.46094	1.22054	0.47294	0.59117	−1.24813	0.63387
		0	2	−0.06603	0.29206	0.16619	0.20774	−0.30286	0.14171
		0	3	0.01312	0.05376	0.05707	0.07133	−0.05564	0.02457
		1	0	0.30736	−0.50949	−0.13088	−0.16360	0.51833	−0.20946
		1	1	0.11710	−0.19714	−0.05779	−0.07224	0.20137	−0.07385
		1	2	0.04579	−0.07889	−0.02601	−0.03252	0.08044	−0.02676
		$i>1$	j	0.00000	0.00000	0.00000	0.00000	0.00000	0.00000

Table 4.2: *Tree level coefficients of the partonic moments without E_l cuts and with a cut $E_l > 1\,\mathrm{GeV}$.*

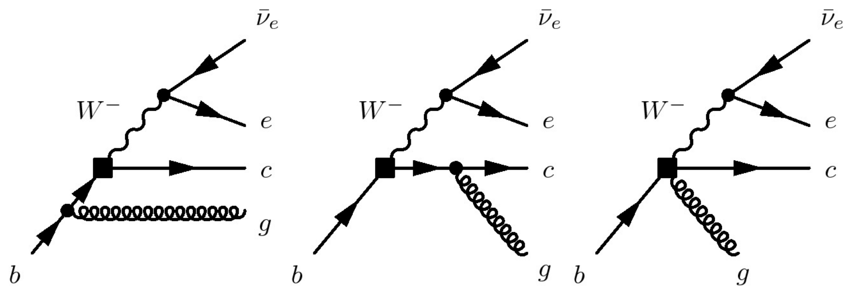


Figure 5.1: *Real Corrections*

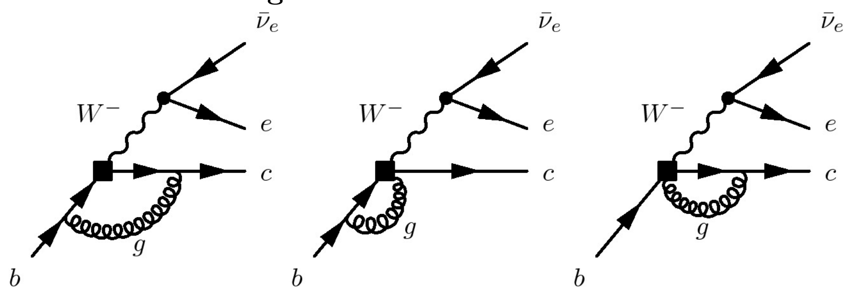


Figure 5.2: *Virtual Corrections*

5 Radiative Corrections

5.1 QCD Corrections

The calculation of the QCD radiative corrections has been performed in [29] and the results in the kinetic scheme have been given in [30] and [18] for the semileptonic moments in the standard model. For the analysis we are going to perform we have also to include the QCD radiative corrections for the inclusive decay with the current (3.81) to order α_s. Thus we have to calculate the Feynman diagrams shown in fig. 5.1 and 5.2 for the real and virtual correction respectively.

The part of the enhanced current with scalar coupling

$$g_L\ \bar{c}\, i\overleftrightarrow{D_\mu} P_- b + g_R\ \bar{c}\, i\overleftrightarrow{D_\mu} P_+ b$$

generates new vertices in form of a quark-quark-gluon-boson vertex as shown in (5.4). They give rise to one new Feynman diagram contributing to the real corrections and two contributing to the virtual corrections as shown in the Feynman diagrams on the right. This is due to the covariant derivative in the definition of the scalar currents, which includes the gluon field:

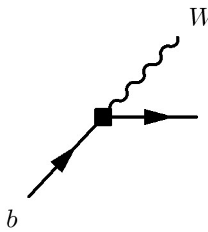
$$D_\mu = \partial_\mu + i g_3 A^a_\mu \lambda_a / 2. \tag{5.1}$$

We introduced the covariant derivative to maintain QCD gauge invariance.

The real and virtual corrections are both IR-divergent. We regulate their IR-divergence with the introduction of a gluon mass which drops out upon summation of the real and virtual corrections being IR-convergent. Calculating the virtual corrections also includes the wave function renormalization of the b and c quark field, given by the counter term.

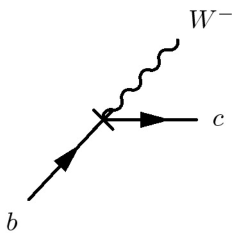
The relevant Feynman rules for the effective vertex prior transition to Fermi interaction for calculating the radiative corrections are

Effective two-quark-W-boson vertex:

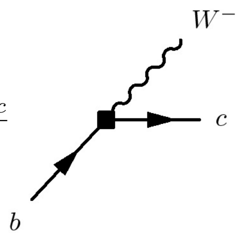


$$= i\frac{eV_{\mathrm{cb}}}{\sqrt{2}s_w}\Big(c_L\gamma_\mu P_- + c_R\gamma_\mu P_+ + g_L p_{b,\mu}P_- + g_R p_{b,\mu}P_+ + d_L(p_b^\mu - p_c^\mu)i\sigma_{\mu\nu}P_- + d_R(p_b^\mu - p_c^\mu)i\sigma_{\mu\nu}P_+\Big) \tag{5.2}$$

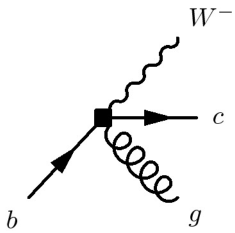
Counter term for the b and c wave function renormalization:



$$= \frac{\delta Z_b + \delta Z_c}{2} \tag{5.3}$$



Effective quark-quark-gluon-boson vertex:



$$= -i\frac{g_s e V_{\mathrm{cb}}}{\sqrt{2}s_w} g_{\mu\nu}\Big(g_L P_- + g_R P_+\Big) \tag{5.4}$$

5.2 Real Corrections

For the calculation of the real and virtual corrections at parton level we follow the method of calculating forward scattering amplitudes and application of cuts via the optical theorem as introduced with the calculation of the non-perturbative corrections. The advantage is that we need not setup a complete new calculation but rather alter the non-perturbative calculation to account for the different process. For the real corrections the forward scattering amplitudes in fig. 5.3 built up the hadronic tensor. The cuts have to be applied in a way to represent a four-particle final state if we include the two leptons described by the unchanged leptonic tensor (4.2). Please note that the hadronic final state now consists of a charm quark and a gluon.

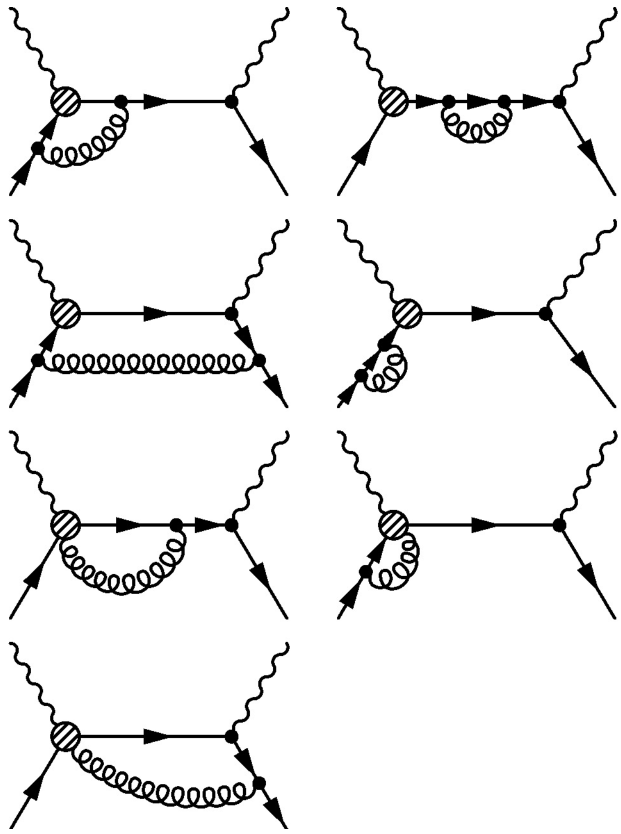


Figure 5.3: *Real corrections as forward scattering amplitudes*

5.2.1 Phase Space Parametrization for the Real Corrections

The phase space of the real corrections is a four-body phase space, which we want to parametrize in the same manner as the three-body phase space of the tree-level process.

The four-body phase space is determined by *five* independent parameters: Four momenta in the final state give $4 \times 3 = 12$. Energy and momentum conservation constrain the parameters with four equations: $12 - 4 = 8$. Due to the arbitrary alignment of the coordinate axis the number of parameters is reduced by 3 to end up with the *five* mentioned above.

We adopt the parametrization from the three-body phase space for the charged lepton and the antineutrino in the rest frame of the decaying b-quark. I. e. we align the charged lepton momentum p_l to the z-axis and the antineutrino momentum p_ν with one angle θ_1 relative to the z-axis in the x-z-plane. The additional gluon in spherical coordinates with the angles θ_1, θ_2 and ϕ. We also include a gluon mass m_g as regulator of the IR-divergences. It will drop out upon addition of the virtual corrections.

$$p_\mathrm{b}^\mu = \begin{pmatrix} m_\mathrm{b} \\ 0 \\ 0 \\ 0 \end{pmatrix} \quad p_\mathrm{l}^\mu = \begin{pmatrix} E_\mathrm{l} \\ 0 \\ 0 \\ E_\mathrm{l} \end{pmatrix} \quad p_\nu^\mu = \begin{pmatrix} E_\nu \\ E_\nu \sin\theta_1 \\ 0 \\ E_\nu \cos\theta_1 \end{pmatrix} \quad p_\mathrm{g}^\mu = \begin{pmatrix} E_\mathrm{g} \\ \sqrt{E_\mathrm{g}^2 - m_\mathrm{g}^2}\, \sin\theta_2 \cos\phi \\ \sqrt{E_\mathrm{g}^2 - m_\mathrm{g}^2}\, \sin\theta_2 \sin\phi \\ \sqrt{E_\mathrm{g}^2 - m_\mathrm{g}^2}\, \cos\theta_2 \end{pmatrix} \quad p_\mathrm{c}^\mu = p_\mathrm{b}^\mu - p_\mathrm{l}^\mu - p_\nu^\mu \tag{5.5}$$

The phase space integral is

$$\begin{aligned}
\mathrm{d}\Pi(3) &= \widetilde{\mathrm{d}p_\mathrm{l}}\,\widetilde{\mathrm{d}p_\nu}\,\widetilde{\mathrm{d}p_\mathrm{g}}\,\widetilde{\mathrm{d}p_{X_\mathrm{c}}}\,(4\pi)^4\delta^4(p_\mathrm{b}-p_\mathrm{l}-p_\nu-p_\mathrm{g}-p_{X_\mathrm{c}}) \\
&= \frac{\mathrm{d}^4p_\mathrm{l}}{(2\pi)^3}\delta(p_\mathrm{l}^2)\Theta(p_{\mathrm{l},0})\,\frac{\mathrm{d}^4p_\nu}{(2\pi)^3}\delta(p_\nu^2)\Theta(p_{\nu,0}) \\
&\quad\times\frac{\mathrm{d}^4p_\mathrm{g}}{(2\pi)^3}\delta(p_\mathrm{g}^2-m_\mathrm{g}^2)\Theta(p_{\mathrm{g},0})\frac{\mathrm{d}^4p_{X_\mathrm{c}}}{(2\pi)^3}\Theta(p_{X_\mathrm{c},0})\,(2\pi)^4\delta^4(p_\mathrm{b}-p_\mathrm{l}-p_\nu-p_\mathrm{g}-p_{X_\mathrm{c}}).
\end{aligned}$$

As with the non-perturbative corrections we do not assume the charm quark to be on-shell in the phase space parametrization, because we will reuse the method of computing a forward matrix element and the on-shellness is taken care of by the imaginary part of the charm quark propagator, which is here the parton level of the background field propagator S_BGF. The gluon propagator is not part of the S_BGF, because it is a hard gluon and the background field is built up from soft gluons.

Integrating over $p_{\mathrm{l},0}$, $p_{\nu,0}$, $p_{\mathrm{g},0}$ and using the four-momentum conservation to integrate over p_{X_c}:

$$\begin{aligned}
&= \frac{\mathrm{d}^3p_\mathrm{l}}{(2\pi)^3 2E_\mathrm{l}}\,\frac{\mathrm{d}^3p_\nu}{(2\pi)^3 2E_\nu}\,\frac{\mathrm{d}^3p_\mathrm{g}}{(2\pi)^3 2E_\mathrm{g}}\bigg|_{p_{X_\mathrm{c}}=p_\mathrm{b}-p_\mathrm{l}-p_\nu-p_\mathrm{g}} \\
&= \frac{\mathrm{d}\Omega_\mathrm{l}\,|\boldsymbol{p}_\mathrm{l}|^2\,\mathrm{d}|\boldsymbol{p}_\mathrm{l}|}{(2\pi)^3 2E_\mathrm{l}}\,\frac{\mathrm{d}\Omega_\nu\,|\boldsymbol{p}_\nu|^2\,\mathrm{d}|\boldsymbol{p}_\nu|}{(2\pi)^3 2E_\nu}\,\frac{\mathrm{d}\Omega_\mathrm{g}\,|\boldsymbol{p}_\mathrm{g}|^2\,\mathrm{d}|\boldsymbol{p}_\mathrm{g}|}{(2\pi)^3 2E_\mathrm{g}}\Bigg|_{p_{X_\mathrm{c}}=p_\mathrm{b}-p_\mathrm{l}-p_\nu-p_\mathrm{g}} \\
&= \frac{4\pi E_\mathrm{l}\,\mathrm{d}E_\mathrm{l}}{2(2\pi)^3}\,\frac{2\pi E_\nu\,\mathrm{d}E_\nu\,\mathrm{d}\cos\theta_1}{2(2\pi)^3}\,\frac{\sqrt{E_\mathrm{g}^2-m_\mathrm{g}^2}\,\mathrm{d}E_\mathrm{g}\,\mathrm{d}\phi\,\mathrm{d}\cos\theta_2}{2(2\pi)^3}\Bigg|_{p_{X_\mathrm{c}}=p_\mathrm{b}-p_\mathrm{l}-p_\nu-p_\mathrm{g}} \qquad (5.6) \\
&= \frac{1}{4(2\pi)^7}E_\mathrm{l}E_\nu\sqrt{E_\mathrm{g}^2-m_\mathrm{g}^2}\,\mathrm{d}E_\mathrm{l}\,\mathrm{d}E_\nu\,\mathrm{d}\cos\theta_1\,\mathrm{d}E_\mathrm{g}\,\mathrm{d}\phi\,\mathrm{d}\cos\theta_2 \qquad (5.7)
\end{aligned}$$

The range of the lepton energy E_l in this decay is[1]

$$0 \le E_\mathrm{l} \le \frac{m_\mathrm{b}}{2}\left(1-\hat{m}_\mathrm{g}-\frac{\rho}{1-\hat{m}_\mathrm{g}}\right) \tag{5.8}$$

with $\hat{m}_\mathrm{g}=m_\mathrm{g}/m_\mathrm{b}$. and the range of the antineutrino energy is determined by the charged lepton energy E_l, the angle θ_1[2]:

$$0 \le E_\nu \le \frac{\frac{m_\mathrm{b}}{2}(1-\rho+\hat{m}_\mathrm{g}^2)-E_\mathrm{l}}{\kappa_1}. \tag{5.9}$$

with $\kappa_1 = \frac{E_\mathrm{l}}{m_\mathrm{b}}\left(1-\sqrt{1-\frac{m_\mathrm{g}^2}{E_\mathrm{g}^2}}\cos\theta_1\right)$.

We use the gluon energy integration to resolve the delta function from the imaginary part of the charm quark propagator, which becomes

$$\mathrm{Im}\left(\frac{1}{p_\mathrm{c}^2-m_\mathrm{c}^2+i\epsilon}\right) = -\frac{\pi}{|2m_\mathrm{b}\kappa_{12}|}\,\delta\!\left(E_\mathrm{g}-\frac{\frac{m_\mathrm{b}}{2}(1-\rho+\hat{m}_\mathrm{g}^2)-E_\mathrm{l}-\kappa_1E_\nu}{\kappa_{12}}\right) \tag{5.10}$$

[1] The upper limit can be derived from squaring the four-momentum conservation $(p_\mathrm{b}-p_\mathrm{l})^2=(p_\nu+p_\mathrm{g}+p_\mathrm{c})^2$ with E_ν=0, $E_\mathrm{g}=m_\mathrm{g}$.

[2] Analogously to E_l the upper limit of E_ν can be derived from $(p_\mathrm{b}-p_\nu)^2=(p_\mathrm{l}+p_\mathrm{g}+p_\mathrm{c})^2$ with E_l from (5.8).

with

$$\kappa_{12} = 1 - (1-\cos\theta_2)\frac{E_{\mathrm{l}}}{m_{\mathrm{b}}} - \left(1 - \sqrt{1 - \frac{m_{\mathrm{g}}^2}{E_{\mathrm{g}}^2}}(\sin\theta_1\sin\theta_2\cos\phi - \cos\theta_1\cos\theta_2)\right)\frac{E_{\nu}}{m_{\mathrm{b}}}.$$

The ranges of the three angle integrations are

$$-1 \le \cos\theta_1, \cos\theta_2 \le 1 \qquad \text{and} \qquad 0 \le \phi \le 2\pi. \tag{5.11}$$

5.2.2 Feynman Diagrams

The leptonic tensor is the same as (4.2):

$$L^{\mu\nu} = 2\left(p_e^{\mu} p_{\nu}^{\nu} - (p_e \cdot p_{\nu}) g^{\mu\nu} + p_e^{\nu} p_{\nu}^{\mu} - i\epsilon^{\alpha\nu\beta\mu} p_{e,\alpha} p_{\nu,\beta}\right) \tag{5.12}$$

and the hadronic tensor is firstly determined by four Feynman diagrams for the forward scattering amplitude due to all possible combinations of squaring initial state radiation (ISR) and final state radiation (FSR). Because of the new quark-quark-gluon-boson vertex there are additional four diagrams contributing as interference terms of the scalar current with the standard-model one. In the following Γ' is the new current except for the gauge term of the covariant derivative which generates the new vertices and thus is separated as the last four diagrams. In the first four diagrams we shall neglect contributions in the new parameters squared. The hadronic tensor is built up by applying the trace formulae (4.34) to the following expression

$$\begin{aligned}
&(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\alpha})\frac{\mathrm{i}(\not p_{\mathrm{b}}-\not p_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\Gamma'^{\mu}\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\Gamma'^{\nu})^{\dagger}\frac{-\mathrm{i}(\not p_{\mathrm{b}}-\not p_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\left(-\mathrm{i}g_{\mathrm{s}}T_b\gamma^{\beta}\right)\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right) \qquad (5.13)\\
&+\Gamma'^{\mu}\frac{\mathrm{i}(\not p_{\mathrm{g}}+\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\alpha})\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\left(-\mathrm{i}g_{\mathrm{s}}T_b\gamma^{\beta}\right)\frac{-\mathrm{i}(\not p_{\mathrm{g}}+\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\Gamma'^{\nu})^{\dagger}\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)\\
&+(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\alpha})\frac{\mathrm{i}(\not p_{\mathrm{b}}-\not p_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\Gamma'^{\mu}\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\left(-\mathrm{i}g_{\mathrm{s}}T_b\gamma^{\beta}\right)\frac{-\mathrm{i}(\not p_{\mathrm{g}}+\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\Gamma'^{\nu})^{\dagger}\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)\\
&+\Gamma'^{\mu}\frac{\mathrm{i}(\not p_{\mathrm{g}}+\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\alpha})\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\Gamma'^{\nu})^{\dagger}\frac{-\mathrm{i}(\not p_{\mathrm{b}}-\not p_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\left(-\mathrm{i}g_{\mathrm{s}}T_b\gamma^{\beta}\right)\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)\\
&-\mathrm{i}g^{\mu\alpha}(g_{\mathrm{L}}P_-+g_{\mathrm{R}}P_+)\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\left(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\beta}\right)\frac{-\mathrm{i}(\not p_{\mathrm{g}}+\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\mathrm{i}c_{\mathrm{L}}\gamma^{\nu}P_-\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)\\
&+\mathrm{i}c_{\mathrm{L}}\gamma^{\mu}P_-\frac{\mathrm{i}(\not p_{\mathrm{g}}+\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\alpha})\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}-\mathrm{i}g^{\nu\beta}(g_{\mathrm{L}}P_-+g_{\mathrm{R}}P_+)\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)\\
&-\mathrm{i}g^{\mu\alpha}(g_{\mathrm{L}}P_-+g_{\mathrm{R}}P_+)\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\mathrm{i}c_{\mathrm{L}}\gamma^{\nu}P_-\frac{-\mathrm{i}(\not p_{\mathrm{b}}-\not p_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\left(\mathrm{i}g_{\mathrm{s}}T_b\gamma^{\beta}\right)\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)\\
&+(\mathrm{i}g_{\mathrm{s}}T_a\gamma^{\alpha})\frac{-\mathrm{i}(\not p_{\mathrm{b}}-\not p_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\mathrm{i}c_{\mathrm{L}}\gamma^{\mu}P_-\frac{\mathrm{i}(\not p_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}-\mathrm{i}g^{\nu\beta}(g_{\mathrm{L}}P_-+g_{\mathrm{R}}P_+)\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)
\end{aligned}$$

This expression does not include any spinors of the bottom quark as external legs, because this is taken care of by the trace formulae. The trace formulae contain the spin-summed spinors in a way to express them in terms of the HQE parameters like m_{B}, $\hat{\mu}_\pi^2$, $\hat{\mu}_{\mathrm{G}}^2$ etc. Due to using the lowest order in the HQE expansion the trace formulae are here applied by multiplying (5.13) by the lowest matrix element from the trace formulae (4.34) and perform the trace. After application of the trace formulae, we take the imaginary part by turning the charm-quark propagator with p_{c} as momentum and the gluon propagator into delta functions which effectively sets these particles on-shell, i. e. $p_{\mathrm{c}}^2 = m_{\mathrm{c}}^2$ and $p_{\mathrm{g}}^2 = m_{\mathrm{g}}^2$. Please note that we use a non-zero gluon mass as regulator of the IR-divergences. The final result obtained by including the virtual corrections must not depend on m_{g}. With the phase space parametrization discussed above the corresponding integrals can be performed. The expressions to integrate are rather lengthy and computed only numerically with a small gluon mass. Again we have to compute the lepton energy moments and the building blocks for the moments of the invariant hadronic mass as in (4.66) and (4.67) respectively. The radiative corrections include both, the real corrections and the virtual corrections and only a combination of both delivers a physical decay rate or moments as needed by the combined fit. Fortunately, the tree-level result and the virtual corrections of the partonic moments with $(\hat{s}_0-\rho)^{>1}$ are zero and thus they can be computed only by evaluating the real corrections, where the gluon mass drops out directly. This is due to the fact, that the integral weight $((\hat{s}_0-\rho))$ is of the same form as the on-shellness relation in the three-body phase space $(p_{\mathrm{c}}^2 - m_{\mathrm{c}}^2)$, which is to be applied for the tree-level and virtual corrections computation. Evaluating the delta function turns the weight identical to zero. Simply more descriptive: the partonic invariant mass is identical to the charm-quark mass squared in the case of the three body decay. The partonic invariant mass distribution is thus a delta function at m_{c}^2, giving zero as mean centered at m_{c}^2 as well as for the variance and skewness. In the real corrections the additional gluon changes the partonic invariant mass. We will present the results together with all other moments at the end of this chapter.

5.3 Renormalization and Running

Once the operators are set up the weak boson can be integrated out, because the process at hand takes place at hadronic scales which are far below the weak boson's mass. This results in non-local operator product previously linked through the weak boson. The operator product expansion yields a series of composite local operators by moving short-distance contributions below a certain scale μ to the Wilson coefficients and consigning the long-distance ones above μ to the local operators. The Wilson coefficients have to be calculated by comparing the series with the "full theory" for equal effect to some required order of precision, which is called "matching". The coefficients and the local operators depend on the introduced scale μ, while the full series must not. The scale for separation μ can be arbitrary, but coinciding with the mass of the heavy degree of freedom integrated out Λ suits best. A change in scale moves contributions from the local operators in the series to the coefficients or vice versa. As a remnant of integrating out the heavy degree of freedom, it's mass will appear in the Wilson coefficients together with the scale of separation μ. In the perturbative expansion in the strong coupling constant α_s both mass scales show up as $\log(\Lambda/\mu)$ in equal power to α_s. With the scales Λ and μ being very disparate, the logarithm will become large and may overwhelm the expansion parameter in downsizing the contributions to the perturbative series order by order. To bypass the problem the operator product expansion has to be performed for $\mu = \Lambda$ with matching the coefficients to the full theory. Then the coefficients are expanded perturbatively with the logarithms being zero and thus not appearing. With help of the renormalization group equation the coefficients can be calculated for a lower scale. The non-perturbative validity of the renormalization group equation sums up all terms of equal power in the logarithms and α_s automatically. This is called *leading-logarithmic approximation (LLA).*

5.3.1 Calculation of the Anomalous Dimension

It is well known that the left- and right-handed currents do not have anomalous dimensions and hence the parts of (3.81) with c_L and c_R are not renormalized. However, the scalar and tensor contributions have anomalous dimensions and hence we need to normalize these operators at some scale and run them down to the scale of the bottom quark.

As usual the renormalization group equation for the running of the Wilson coefficient can be obtained from the requirement that the physical matrix elements must not depend on the arbitrary scale μ:

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\langle c\ell\nu_e|\mathcal{H}_{\mathrm{eff}}|b\rangle \tag{5.14}$$

By application of the OPE the Hamiltonian can be written as a linear combination of the Wilson coefficients and the operator basis:

$$\begin{aligned}\langle c\ell\nu_e|\mathcal{H}_{\mathrm{eff}}|b\rangle \;=\; & \frac{4G_{\mathrm{F}}V_{\mathrm{cb}}}{\sqrt{2}}\cdot\langle c\ell\nu_e|\left[c_L(\bar{c}\,\gamma_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) + c_R(\bar{c}\,\gamma_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e)\right]|b\rangle \\ & + \frac{4G_{\mathrm{F}}V_{\mathrm{cb}}}{\sqrt{2}}\boldsymbol{C}\cdot\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle,\end{aligned} \tag{5.15}$$

where we separated the left- and right-handed vector currents with vanishing anomalous dimension and mass dimension 6 from all other operators having mass dimension 7:

$$\boldsymbol{C} = \begin{pmatrix} g_L \\ g_R \\ d_L \\ d_R \\ c_L^{m_b} \\ c_R^{m_b} \\ c_L^{m_c} \\ c_R^{m_c} \end{pmatrix} \qquad \boldsymbol{\mathcal{O}} = \begin{pmatrix} (\bar{c}\,i\overleftrightarrow{D}_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (\bar{c}\,i\overleftrightarrow{D}_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (i\partial^\nu(\bar{c}\,i\sigma_{\mu\nu}P_- b))(\bar{e}\,\gamma^\mu P_- v_e) \\ (i\partial^\nu(\bar{c}\,i\sigma_{\mu\nu}P_+ b))(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_b\,\bar{c}\,\gamma_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_b\,\bar{c}\,\gamma_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_c\,\bar{c}\,\gamma_\mu P_- b)(\bar{e}\,\gamma^\mu P_- v_e) \\ (m_c\,\bar{c}\,\gamma_\mu P_+ b)(\bar{e}\,\gamma^\mu P_- v_e) \end{pmatrix}. \tag{5.16}$$

Thus the coefficients $\boldsymbol{C}$ contain the short-distance contributions, i. e. the physics above a scale μ, and the matrix elements $\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle$ contain the long-distance contributions from below the scale μ. The last four operators and coefficients in (5.16) do not appear in the effective field theory derivation, but they are induced by the operator mixing we are about to explain. Their absence in the effective field theory derivation turns into a boundary condition for the renormalization group equation at the high scale Λ.

The renormalization of the matrix element consists of the wave function renormalization for the b quark (Z_b) and the c quark field (Z_c) and the renormalization of the Wilson coefficients $\boldsymbol{C}^{(0)} = Z^C\boldsymbol{C}$. For the operators induced by mixing the mass renormalization for m_b and m_c has to be taken into account, too, which we will not write down in the following discussion. As the renormalization is multiplicative the renormalization of the Wilson coefficients is equivalent to renormalizing the operator vector $\boldsymbol{\mathcal{O}}$ with $(Z_{\mathcal{O}})$:

$$\boldsymbol{C}^{(0)}\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}(b^{(0)},c^{(0)})|b\rangle = \frac{Z^C}{\sqrt{Z_b Z_c}}\boldsymbol{C}\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle.$$

The Wilson coefficient renormalization constant is a 6×6 matrix containing the Wilson coefficients renormalization constants and describing the operator mixing as a rotation in the space the operator basis spans for a certain scale Λ. The renormalization condition (5.14) with bare quantities reads

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\Big(\boldsymbol{C}\cdot\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle\Big) = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\Big(\frac{1}{Z^C}\,\boldsymbol{C}^{(0)}\cdot\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle\Big) \tag{5.17}$$

$$= -\frac{1}{(Z^C)^2}\frac{\mathrm{d}Z^C}{\mathrm{d}\ln\mu}\,\boldsymbol{C}^{(0)}\cdot\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle + \frac{1}{Z^C}\,\boldsymbol{C}^{(0)}\cdot\frac{\mathrm{d}}{\mathrm{d}\ln\mu}\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle \tag{5.18}$$

$$\Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle = \underbrace{-\frac{1}{Z^C}\,\frac{\mathrm{d}Z^C}{\mathrm{d}\ln\mu}}_{\gamma(\mu)}\,\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle. \tag{5.19}$$

$$\Longleftrightarrow \qquad \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle = \gamma(\mu)\,\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle. \tag{5.20}$$

The logarithmic derivative of the Operator $\boldsymbol{\mathcal{O}}$ defines the anomalous-dimension matrix γ which describes the linear operator mixing for an infinitesimal change in $\log\mu$.

The μ-independence of the bare Wilson coefficients $\boldsymbol{C}^{(0)}$

$$0 = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}\boldsymbol{C}^{(0)} = \frac{\mathrm{d}}{\mathrm{d}\ln\mu}(Z^C\boldsymbol{C}) = \boldsymbol{C}\frac{\mathrm{d}Z^C}{\mathrm{d}\ln\mu} + Z^C\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}\ln\mu} \tag{5.21}$$

$$\Longleftrightarrow \qquad \frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}\ln\mu} = \underbrace{-\frac{1}{Z^C}\,\frac{\mathrm{d}Z^C}{\mathrm{d}\ln\mu}}_{\gamma^T(\mu)}\,\boldsymbol{C} \tag{5.22}$$

leads to their renormalization group equation:

$$\frac{\mathrm{d}\boldsymbol{C}}{\mathrm{d}\ln\mu} = \gamma^T(\mu)\,\boldsymbol{C} \tag{5.23}$$

According to the definition of the anomalous-dimension matrix for the matrix element the transpose of the matrix has to be used in the renormalization group equation for the Wilson coefficients.

In the intended $\overline{\mathrm{MS}}$ renormalization scheme the constants Z^C are chosen to cancel the pure pole divergences $1/\varepsilon^n$ and no finite parts. Hence we can expand Z^C in its Laurent series in ε to first order:

$$Z^C = 1 + \frac{Z_1^C}{\varepsilon}.$$

From the definition of the anomalous dimension we can derive

$$\gamma = -\frac{1}{Z^C}\frac{\mathrm{d}Z^C}{\mathrm{d}\ln\mu} = -\frac{1}{\varepsilon}\frac{\mathrm{d}Z_1^C}{\mathrm{d}\ln\mu} = -\frac{2g}{\varepsilon}\,\underbrace{\frac{\partial g}{\partial\ln\mu}}_{\beta(\varepsilon,g(\mu))}\,\frac{\partial Z_1^C}{\partial g^2} = 2\alpha_s\frac{\partial Z_1^C}{\partial\alpha_s},$$

where we used $\beta(\varepsilon,g(\mu)) = -\varepsilon g + \beta(g)$ taking only the leading part in $1/\varepsilon$ and $g^2 = 4\pi\alpha_s$.

The one-loop renormalization constants Z_1^C are in the $\overline{\text{MS}}$ scheme the divergent part of $\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle$ and can thus be obtained by extracting the divergent parts of the one-loop amplitude of $\langle c\ell\nu_e|\boldsymbol{\mathcal{O}}|b\rangle$. Technically this can be done by using the result of the virtual corrections including all counter terms, determine their UV-divergent parts, and identify the Dirac structures were the parameters g_L, g_R, d_L, d_R, $c_L^{m_b}$, $c_R^{m_b}$, $c_L^{m_c}$ and $c_R^{m_c}$ show up in the one-loop result, indicating the mixture between the parameters' dedicated Dirac-structures as of (3.81) and the identified ones.

The anomalous dimension matrix for the Wilson coefficients yields

$$\gamma^T(\mu) = \frac{2\alpha_s(\mu)}{3\pi}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 & 0 \\ 0 & \mathbf{1} & 0 & \mathbf{1} & 0 & 0 & 0 & 0 \\ 0 & \mathbf{3} & 0 & 0 & \mathbf{3} & 0 & 0 & 0 \\ \mathbf{3} & 0 & 0 & 0 & 0 & \mathbf{3} & 0 & 0 \\ \mathbf{3} & 0 & 0 & 0 & 0 & 0 & \mathbf{3} & 0 \\ 0 & \mathbf{3} & 0 & 0 & 0 & 0 & 0 & \mathbf{3} \end{pmatrix}. \tag{5.24}$$

5.3.2 Running of the Wilson Coefficients

Assuming massless QCD the renormalization group equation of the Wilson coefficients (5.23) can be rewritten due to renormalization group flow only induced by the running of the strong coupling constant. The anomalous dimension matrix then depends on the scale μ only through α_s:

$$\gamma(\mu) = \gamma\big(\alpha_s(\mu)\big)$$

The total $\log\mu$-derivative must be replaced by

$$\frac{\mathrm{d}}{\mathrm{d}\ln\mu} = \frac{\partial}{\partial\ln\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s},$$

where we used the definition of the β-function $\mathrm{d}\alpha_s(\mu)/\mathrm{d}\mu = \beta\big(\alpha_s(\mu)\big)$. The renormalization group equation for the Wilson coefficient becomes

$$\left(\frac{\partial}{\partial\ln\mu} + \beta(\alpha_s)\frac{\partial}{\partial\alpha_s}\right)\boldsymbol{C} = \gamma^T\big(\alpha_s(\mu)\big)\,\boldsymbol{C}. \tag{5.25}$$

For the one-loop level performed in this analysis we only take the leading term of the β-function's α_s-expansion $\beta_0\alpha_s^2/(4\pi)$ with $\beta_0 = -2/3(33-2n_f)$ and $n_f = 5$, the number of active quark flavours at the scale μ:

$$\beta_0\frac{\alpha_s^2}{4\pi}\frac{\partial}{\partial\alpha_s}\boldsymbol{C} = \gamma^T\big(\alpha_s(\mu)\big)\,\boldsymbol{C}. \tag{5.26}$$

We seek a solution of this equation with the initial conditions

$$c_{L/R}^{m_b}(\Lambda) = 0 = c_{L/R}^{m_c}(\Lambda)\,, \tag{5.27}$$

which yields the Wilson coefficients' running from the scale Λ to the interaction scale μ:

$$\begin{aligned}
c_{L/R}(\mu) &= c_{L/R}(\Lambda) \\
g_{L/R}(\mu) &= g_{L/R}(\Lambda) \\
d_{L/R}(\mu) &= \left(g_{L/R}(\Lambda) + d_{L/R}(\Lambda)\right)\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)}\right)^{\frac{4}{3\beta_0}} - g_{L/R}(\Lambda) \\
c^{m_b}_{L/R}(\mu) &= g_{R/L}(\Lambda)\left(\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)}\right)^{\frac{4}{\beta_0}} - 1\right) \\
c^{m_c}_{L/R}(\mu) &= g_{L/R}(\Lambda)\left(\left(\frac{\alpha_s(\Lambda)}{\alpha_s(\mu)}\right)^{\frac{4}{\beta_0}} - 1\right),
\end{aligned} \tag{5.28}$$

with the leading order solution of the renormalization group equation for the coupling α_s, which can be expressed in the form

$$\alpha_s(\Lambda) = \frac{\alpha_s(\mu)}{1 + \beta_0 \frac{\alpha_s(\mu)}{2\pi} \log \frac{\Lambda}{\mu}} \tag{5.29}$$

we may rewrite (5.28) as

$$\begin{aligned}
c_{L/R}(\mu) &= c_{L/R}(\Lambda) \\
g_{L/R}(\mu) &= g_{L/R}(\Lambda) \\
d_{L/R}(\mu) &= \left(g_{L/R}(\Lambda) + d_{L/R}(\Lambda)\right)\left(1 + \beta_0 \frac{\alpha_s(\mu)}{2\pi} \log \frac{\Lambda}{\mu}\right)^{-\frac{4}{3\beta_0}} - g_{L/R}(\Lambda) \\
c^{m_b}_{L/R}(\mu) &= g_{R/L}(\Lambda))\left(\left(1 + \beta_0 \frac{\alpha_s(\mu)}{2\pi} \log \frac{\Lambda}{\mu}\right)^{-\frac{4}{\beta_0}} - 1\right) \\
c^{m_c}_{L/R}(\mu) &= g_{L/R}(\Lambda))\left(\left(1 + \beta_0 \frac{\alpha_s(\mu)}{2\pi} \log \frac{\Lambda}{\mu}\right)^{-\frac{4}{\beta_0}} - 1\right)
\end{aligned} \tag{5.30}$$

This includes the logarithmic corrections to all orders in $\alpha_s \log(\Lambda/\mu)$.

Expanding (5.30) to first order in α_s we can explicitly check the cancellation of the μ-dependence of the amplitude with

$$\begin{aligned}
c_{L/R}(\mu) &= c_{L/R}(\Lambda) \\
g_{L/R}(\mu) &= g_{L/R}(\Lambda) \\
d_{L/R}(\mu) &= d_{L/R}(\Lambda) - \left(g_{L/R}(\Lambda) + d_{L/R}(\Lambda)\right)\frac{2\alpha_s(\mu)}{3\pi} \log\left(\frac{\Lambda}{\mu}\right) \\
c^{m_b}_{L/R}(\mu) &= -g_{R/L}(\Lambda)\frac{2\alpha_s(\mu)}{\pi} \log\left(\frac{\Lambda}{\mu}\right) \\
c^{m_c}_{L/R}(\mu) &= -g_{L/R}(\Lambda)\frac{2\alpha_s(\mu)}{\pi} \log\left(\frac{\Lambda}{\mu}\right).
\end{aligned} \tag{5.31}$$

This can be seen by substituting (5.31) in the tree-level amplitude and adding the one-loop expression. The logarithms from (5.31) cancel the corresponding logarithms appearing in the one-loop amplitude directly.

Expanding coefficients c_i in general in a perturbative series at some low scale μ leads to

$$\begin{aligned}
c_i(\Lambda/\mu,\alpha_s) = \quad & b_i^{00} \\
& + b_i^{11}\left(\frac{\alpha_s}{4\pi}\right)\log\frac{\Lambda}{\mu} \quad + b_i^{10}\left(\frac{\alpha_s}{4\pi}\right) \\
& + b_i^{22}\left(\frac{\alpha_s}{4\pi}\right)^2\log^2\frac{\Lambda}{\mu} \quad + b_i^{21}\left(\frac{\alpha_s}{4\pi}\right)^2\log\frac{\Lambda}{\mu} \quad + b_i^{20}\left(\frac{\alpha_s}{4\pi}\right)^2 \\
& + b_i^{33}\left(\frac{\alpha_s}{4\pi}\right)^3\log^3\frac{\Lambda}{\mu} \quad + b_i^{32}\left(\frac{\alpha_s}{4\pi}\right)^3\log^2\frac{\Lambda}{\mu} \quad + b_i^{31}\left(\frac{\alpha_s}{4\pi}\right)^3\log\frac{\Lambda}{\mu} \quad + b_i^{30}\left(\frac{\alpha_s}{4\pi}\right)^3 \\
& + \dots
\end{aligned} \tag{5.32}$$

At the high scale Λ the logarithms vanish and the expansion becomes

$$c_i(\Lambda/\mu = 1,\alpha_s) = \sum_n a_i^{(n)}\left(\frac{\alpha_s}{4\pi}\right) \tag{5.33}$$

Performing the expansion at the high scale Λ and using the renormalization group equation to express the series at the lower scale μ effectively sums up the columns in (5.32) depending on the order in α_s of (5.33). This resummation of "leading logarithms" can be seen by comparing (5.30) with the first column of (5.32). The first order expansion (5.31) thus gives the terms for b_i^{00} and b_i^{11}.

In this analysis we perform a complete one loop calculation at the low scale. As a by-product we obtain the one loop anomalous dimensions of the coefficients in order to resum the leading logarithms for the interaction/low scale, i.e. the first column of (5.32). This is done by exchanging the tree level coefficients at the low scale μ with the ones at the high scale according to (5.28).

The one loop calculation yields also b_i^{10} additionally to the first column, but the second column of (5.32) could only be resummed by calculating the two loop anomalous dimensions which is beyond the scope of this analysis. Rather we shall fix this scale to be $\mu = m_b$, which is the relevant scale of the decay process, assuming that the full NLO calculation would fix a scale of this order.

The advantage of this procedure is that the kinematic effects, which lead to a distortion of the spectra and thus have an impact on the moments, are given by these finite terms. We expect that a full NLO calculation will lead to very similar results.

5.4 Virtual Corrections

Computing the virtual corrections in the manner of a forward scattering amplitude includes six Feynman diagrams and two corresponding counter terms. Only the interference term of the virtual corrections with tree-level is of order α_s. In the standard model this amounts for two diagrams plus counterterms, instead of six, which is due to the new quark-quark-gluon-boson vertex from the scalar current. The leptonic tensor is unchanged and the phase space is the same as for the non-perturbative corrections, because the gluon is only virtual and the cuts are only applied to single charm-quark propagators as in the tree level case.

As we need the additional four vector currents with quark masses it is convenient for the discussion to introduce instead of the enhanced current a vector similar to $\mathcal{O}$, but without the leptonic part (indicated by the hat) and not the gauge part of the covariant derivative (indicated by the prime as used with the real corrections):

$$\hat{\boldsymbol{\mathcal{O}}}' = \begin{pmatrix} (\bar{c}\, i\overleftrightarrow{\partial_\mu} P_- b) \\ (\bar{c}\, i\overleftrightarrow{\partial_\mu} P_+ b) \\ (i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_- b)) \\ (i\partial^\nu(\bar{c}\, i\sigma_{\mu\nu} P_+ b)) \\ (m_b\, \bar{c}\, \gamma_\mu P_- b) \\ (m_b\, \bar{c}\, \gamma_\mu P_+ b) \\ (m_c\, \bar{c}\, \gamma_\mu P_- b) \\ (m_c\, \bar{c}\, \gamma_\mu P_+ b) \end{pmatrix} . \tag{5.34}$$

Now we can define a corresponding vector of renormalization constants and parameters:

$$\widehat{\boldsymbol{\delta Z}}' = \begin{pmatrix} \frac{1}{2}(\delta Z_{\mathrm{b,L}} + \delta Z_{\mathrm{c,L}})\, c_{\mathrm{L}} \\ \frac{1}{2}(\delta Z_{\mathrm{b,R}} + \delta Z_{\mathrm{c,R}})\, c_{\mathrm{R}} \\ \frac{1}{2}(\delta Z_{\mathrm{b,L}} + \delta Z_{\mathrm{c,L}})\, d_{\mathrm{L}} \\ \frac{1}{2}(\delta Z_{\mathrm{b,R}} + \delta Z_{\mathrm{c,R}})\, d_{\mathrm{R}} \\ \left(\frac{1}{2}(\delta Z_{\mathrm{b,L}} + \delta Z_{\mathrm{c,L}}) + \frac{1}{m_{\mathrm{b}}}\delta Z_{m_{\mathrm{b}}}\right) c_{\mathrm{L}}^{m_{\mathrm{b}}} \\ \left(\frac{1}{2}(\delta Z_{\mathrm{b,R}} + \delta Z_{\mathrm{c,R}}) + \frac{1}{m_{\mathrm{b}}}\delta Z_{m_{\mathrm{b}}}\right) c_{\mathrm{R}}^{m_{\mathrm{b}}} \\ \left(\frac{1}{2}(\delta Z_{\mathrm{b,L}} + \delta Z_{\mathrm{c,L}}) + \frac{1}{m_{\mathrm{b}}}\delta Z_{m_{\mathrm{c}}}\right) c_{\mathrm{L}}^{m_{\mathrm{c}}} \\ \left(\frac{1}{2}(\delta Z_{\mathrm{b,R}} + \delta Z_{\mathrm{c,R}}) + \frac{1}{m_{\mathrm{b}}}\delta Z_{m_{\mathrm{c}}}\right) c_{\mathrm{R}}^{m_{\mathrm{c}}} \end{pmatrix} . \tag{5.35}$$

where $Z_{\mathrm{b,L}}$, $Z_{\mathrm{b,R}}$ are the left-handed bottom quark and charm quark wave function renormalization constant respectively and $Z_{\mathrm{c,L}}$, $Z_{\mathrm{c,R}}$ the corresponding renormalization constant for the charm quark wave function. The quantities $\delta Z_{m_{\mathrm{b}}}$ and $\delta Z_{m_{\mathrm{c}}}$ are the mass renormalization constants for the bottom quark and charm quark mass.

The hadronic tensor can be found by application of the trace formulae to the expression

$$(\mathrm{i}g_{\mathrm{s}}T_a\gamma^\alpha)\,\frac{\mathrm{i}(\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}\boldsymbol{C}\cdot\hat{\boldsymbol{\mathcal{O}}}'^\mu\,\frac{-\mathrm{i}(\not{p}_{\mathrm{g}}+\not{p}_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\left(\mathrm{i}g_{\mathrm{s}}T_b\gamma^\beta\right)\frac{-\mathrm{i}(\not{p}_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\boldsymbol{C}\cdot\hat{\boldsymbol{\mathcal{O}}}'^\nu\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)+\mathrm{h.c.} \tag{5.36}$$

$$+(\mathrm{i}g_{\mathrm{s}}T_a\gamma^\alpha)\,\frac{\mathrm{i}(\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}+m_{\mathrm{b}})}{(p_{\mathrm{b}}-p_{\mathrm{g}})^2-m_{\mathrm{b}}^2}-\mathrm{i}g^{\mu\beta}(g_{\mathrm{L}}P_-+g_{\mathrm{R}}P_+)\frac{-\mathrm{i}(\not{p}_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\mathrm{i}c_{\mathrm{L}}\gamma^\nu P_-\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)+\mathrm{h.c.} \tag{5.37}$$

$$+-\mathrm{i}g^{\mu\alpha}(g_{\mathrm{L}}P_-+g_{\mathrm{R}}P_+)\frac{\mathrm{i}(\not{p}_{\mathrm{g}}+\not{p}_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{g}}+p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}(\mathrm{i}g_{\mathrm{s}}T_a\gamma^\alpha)\,\mathrm{i}c_{\mathrm{L}}\gamma^\nu P_-\left(-\mathrm{i}\frac{\delta_{ab}g_{\alpha\beta}}{p_{\mathrm{g}}^2}\right)+\mathrm{h.c.} \tag{5.38}$$

$$+\widehat{\boldsymbol{\delta Z}}'\cdot\hat{\boldsymbol{\mathcal{O}}}'^\mu\,\frac{\mathrm{i}(\not{p}_{\mathrm{c}}+m_{\mathrm{c}})}{(p_{\mathrm{c}})^2-m_{\mathrm{c}}^2}\boldsymbol{C}\cdot\hat{\boldsymbol{\mathcal{O}}}'^\nu \tag{5.39}$$

5.4.1 Counter Terms

The counter terms subtract the UV-divergent part of the loop diagrams. For the counter terms of the vertex correction it turns out that only the wave function renormalization of the bottom- and charm-quark field is needed. The counter term of the one-loop correction to the new vertices contains only these wave function renormalization as well. As discussed in the section about renormalization an running of the operators, we have to include the virtual corrections of the left- and right-handed vector currents with either the bottom-quark or charm-quark mass. As the quark masses are scale dependent their mass renormalization has to be included for the left- and right-handed vector currents with quark masses. We discussed the counterterms in 2.3.2.

5.4.2 N-Points Integrals

In the calculation of the virtual corrections so-called N-point functions arise. In loop diagrams the number of momentum integrations determines the loop level. In the present one-loop process standard integrals in one variable appear that are known as N-point functions, depending on the number of propagator denominators including the integration variable. Here only the scalar one-, two- and three-point functions (A_0, B_0, C_0) emerge that are defined as follows:

$$A_0(m^2) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \frac{\mathrm{d}^D q}{q^2 - m^2} \tag{5.40}$$

$$B_0(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \frac{\mathrm{d}^D q}{\left(q^2 - m_1^2\right)\left((q+p)^2 - m_2^2\right)} \tag{5.41}$$

$$C_0(p_1^2, p_2^2, (p_1+p_2)^2, m_1^2, m_2^2, m_3^2) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \frac{\mathrm{d}^D q}{\left(q^2 - m_1^2\right)\left((q+p_1)^2 - m_2^2\right)\left((q+p_1+p_2)^2 - m_3^2\right)} \tag{5.42}$$

The calculation of these integrals is done with a technique called dimensional regularization. Some of the integrals are divergent in four dimensions. They can be regularized by computing them in $D = 4 - \epsilon$ dimensions, separating the UV divergences as poles in ϵ. Using renormalized field, masses and coupling, we have to include counter terms which compensate these poles in ϵ, yielding a divergence-free result. Infrared divergences are here addressed by introducing an unphysical gluon mass, which shall drop out upon joining real and virtual corrections. They, as well, could be handled by dimensional regularization as an alternative method.

The scalar functions are subtypes of more general ones with the integration momentum in the numerator, like

$$B^\mu(p^2, m_1^2, m_2^2) = \frac{(2\pi\mu)^{4-D}}{\mathrm{i}\pi^2} \int \frac{q^\mu\, \mathrm{d}^D q}{\left(q^2 - m_1^2\right)\left((q+p)^2 - m_2^2\right)}$$

but all of these can be expressed in scalar functions. This is done by decomposing them into covariants as

$$B^\mu = p^\mu B_1$$

and contracting the equation with outer momenta (here only p) and the metric $g^{\mu\nu}$ in all possible ways yielding a linear equation system between scalar functions and the tensor function, which can be solved to the tensor coefficients. Here only a contraction with p^μ is needed yielding directly

$$B_1(p^2,m_1^2,m_2^2) = \frac{1}{2p^2}\left(A_0(m_1^2) - A_0(m_2^2) - (p^2 - m_2^2 + m_1^2)B_0(p^2,m_1^2,m_2^2)\right) \tag{5.43}$$

The identification of N-point functions and their decomposition into scalar functions can be automatized and is e. g. performed by the `MATHEMATICA` package `FeynCalc` [31]. The final numerical evaluation of the scalar integrals is done by the programming package `LoopTools` [32], which has `MATHEMATICA`, `FORTRAN` and `C/C++` interfaces. `LoopTools` is based on the `FF` [33] package.

5.5 Results of the Radiative Corrections

The radiative corrections consist of the real and virtual corrections which have to be added in order to cancel the IR-divergences. The UV-divergences are compensated by the renormalization addressed by including counter terms into the virtual correction calculation. Furthermore, the new operators have a nontrivial scale behavior and mix under each other and even into operators that are not assumed to appear at the renormalization scale. As we include the radiative corrections the mixing in form of the coefficients replacement (5.28) has to be included on all orders for the computed moments. Therefore we present the non-perturbative results here again, but accounting for the operator mixing and thus we show results as coefficients of the parameters at the high scale, i. e. $g_{\rm L}(\Lambda)$, $g_{\rm R}(\Lambda)$, $d_{\rm L}(\Lambda)$,$d_{\rm R}(\Lambda)$. Table 5.1 we list all needed moments of the lepton energy spectrum with and without a lower cut on the lepton energy of 1 GeV and in table 5.2 all building blocks for the moments of the hadronic invariant mass without a cut and in 5.3 with a lower lepton energy cut of 1 GeV, which have been split due to their bulkiness. In tables 5.4 and 5.5 the summed up tree level and α_s/π coefficients of the leptonic moments and hadronic moments respectively without a cut on the lepton energy for $\mu = 2.3, 4.6$ and $9.2\,\mathrm{GeV}$ are shown to demonstrate the scale dependence and the overall sensitivity of the various coupling parameters.

The lepton energy moments are defined the same way as in the non-perturbative corrections:

$$L_n = \frac{\langle \hat{E}_{\rm l}^n \rangle}{\Gamma_0} = \frac{1}{\Gamma_0}\int_{E_{\rm cut}} \mathrm{d}E_{\rm l} \int \mathrm{d}E_{\nu} \int \mathrm{d}\cos\Theta\ \hat{E}_{\rm l}^n \frac{\mathrm{d}\Gamma}{\mathrm{d}E_{\rm l}\mathrm{d}E_{\nu}\,\mathrm{d}\cos\Theta} \tag{5.44}$$

and the building blocks for the moments of the hadronic invariant mass

$$H_{ij} = \frac{\langle (\hat{\mathrm{s}}_0 - \rho)^i \hat{E}_0^j \rangle}{\Gamma_0} = \frac{1}{\Gamma_0}\int_{E_{\rm cut}} \mathrm{d}E_{\rm l} \int \mathrm{d}E_{\nu} \int \mathrm{d}\cos\Theta\ (\hat{s}_0 - \rho)^i\, \hat{E}_0^j \frac{\mathrm{d}\Gamma}{\mathrm{d}E_{\rm l}\mathrm{d}E_{\nu}\,\mathrm{d}\cos\Theta} \tag{5.45}$$

change in so far as the partonic energy $\hat{E}_0$ and partonic invariant mass $\hat{s}_0$ now include the gluon in the case of the real corrections, as being a part of the hadronic system:

$$\hat{E}_0 = \frac{1}{m_{\rm b}}(p_{c,0} + p_{g,0}) \tag{5.46}$$

$$\hat{s}_0 = \frac{1}{m_{\rm b}}(p_{\rm c} + p_{\rm g})^2. \tag{5.47}$$

The moments have additionally to the expansion in $1/m_b$ (4.71) an expansion in α_s:

$$
\begin{aligned}
L_n^{c_1 c_2} &= L_n^{(c_1c_2;\, m_b^0, \alpha_s^0)} + \frac{\mu_\pi^2}{m_b^2} L_n^{(c_1c_2;\, m_b^2, \alpha_s^0)} + \frac{\mu_g^2}{3\, m_b^2} L_n^{(c_1c_2;\, m_b^2, \alpha_s^0)} + \frac{\alpha_s}{\pi} L_n^{(c_1c_2;\, m_b^0, \alpha_s^1)} + \cdots \\
H_{ij}^{c_1 c_2} &= H_{ij}^{(c_1c_2;\, m_b^0, \alpha_s^0)} + \frac{\mu_\pi^2}{m_b^2} H_{ij}^{(c_1c_2;\, m_b^2, \alpha_s^0)} + \frac{\mu_g^2}{3\, m_b^2} H_{ij}^{(c_1c_2;\, m_b^2, \alpha_s^0)} + \frac{\alpha_s}{\pi} H_{ij}^{(c_1c_2;\, m_b^0, \alpha_s^1)} + \cdots
\end{aligned}
$$

In tab. 5.1, 5.2 and 5.3 we present the perturbative contributions at $\mathcal{O}(\alpha_s)$ and the non-perturbative contributions up to $\mathcal{O}(1/m_{\mathrm{b}}^2)$ of all moments needed for the combined fit. For the numerical analysis we use $m_b^{\mathrm{kin}}(1\,\mathrm{GeV}) = 4.6\,\mathrm{GeV}$ and $\rho = m_c^2/m_b^2 = 0.0625$.

5.5.1 Mass Scheme

In the present analysis we will use the kinetic mass scheme, which we already discussed in section 2.3.2. The ratio $\varrho = m_{\mathrm{c}}^2/m_{\mathrm{b}}^2$ is the same in both schemes to the needed one loop level. Thus the scheme dependence of the rate only comes through the m_{b}^5 term in the common prefactor and will only correct the tree level result as for the one loop result the correction will be of $\mathcal{O}(\alpha_s^2)$. In the end the results will be normalized to the tree level rate with the kinetic mass $m_{\mathrm{b}}^{\mathrm{kin}}$. So the $\mathcal{O}(\alpha_s)$ result gains contributions from the normalized tree level rate due to the change of the mass scheme for the m_{b}^5 term with the factor from (2.80) at $\mu_f = 1\,\mathrm{GeV}$ expanded in its Taylor series to the first order in α_s with $m_{\mathrm{b}} = 4.6\,\mathrm{GeV}$:

$$
\left(\frac{m_{\mathrm{q}}^{\mathrm{pole}}}{m_{\mathrm{q}}^{\mathrm{kin}}(1\,\mathrm{GeV})}\right)^5 \approx 1 + \frac{20}{3}\frac{\alpha_s}{\pi}\left(\frac{4}{3}\frac{\mathrm{GeV}}{m_{\mathrm{b}}} + \frac{(\mathrm{GeV})^2}{2m_{\mathrm{b}}^2}\right) \approx 1 + 2.0899\,\frac{\alpha_s}{\pi}. \tag{5.48}
$$

5.6 Discussion of the results

It turns out that the radiative corrections to the scalar and tensor admixtures are sizable, i. e. the α_s/π coefficients are large. In addition, these coefficients have the opposite sign as the tree level piece, and hence a substantial reduction of the tree result is expected.

Table 5.4 contains the sum of the tree level and the α_s contributions using the one-loop expression for the running coupling α_s. As discussed above, the full NLO expressions for the scalar and tensor couplings are not available yet and hence a residual scale dependence remains. We expect the scale to be of the order of m_b and hence we evaluate the expressions for $\mu = m_b/2$, m_b and $2m_b$. For c_L^2 as well as for $c_L c_R$ the scale dependence is weak and originates from yet unknown NNLO effects. Due to the large α_s/π coefficients the scale dependence for the tensor couplings is sizable, while it is huge for the scalar couplings, since the tree contribution is almost cancelled by the radiative correction. A full NLO calculation will very likely not improve this situation and hence we have to conclude that we will not have a good sensitivity to the tensor couplings and practically no sensitivity to the scalar couplings, at least for the lepton-energy moments.

The coefficients of the nonperturbative contributions at tree level are in general of similar size as the ones of the α_s corrections. Since $\alpha_s/\pi \sim \mu_\pi/m_b^2$, the non-perturbative corrections are of similar importance as the radiative ones. However, the leptonic moments are all dominated by the tree-level contribution and hence the radiative as well as the nonperturative corrections to the moments are small.

Tables 5.3 and 5.2 contain the various hadronic moments computed without and with a cut on the lepton energy. For the $i = 0$ moments we have to draw the same conclusion as for the leptonic moments: The scalar and tensor couplings have large and opposite-sign coefficients compared to the tree level piece; this leads in the same way to a sizable reduction of the tree level result as well as to a large scale dependence, which is shown in table 5.5, where the result up to order α_s is shown.

Clearly the moments with $i > 0$ do not have a tree level contribution at the partonic level since the tree-level partonic rate is proportional to the mass shell delta function $\delta(\hat{s}_0 - \rho)$. For these moments the leading contributions are at order α_s or $1/m_b^2$. Hence their dependence on the scale is given by the dependence of α_s. However, here the radiative corrections are small compared to the non-perturbative ones. The non-perturbative corrections at tree level contain also derivatives of the mass shell delta function $\delta(\hat{s}_0 - \rho)$, where at leading order $1/m_b$ the maximum number of derivatives is two. Due to this, the first and second i moments are of order $1/m_b^2$; higher moments with $i > 2$ will only have contributions of order $1/m_b^3$ or higher.

The sensitivity to a possible new-physics contribution is mainly limited by the precision of the standard-model calculation. Current analyses use up to the second moments in both the leptonic energy and the invariant mass squared. The highest moments included in the standard-model analyses are (roughly) sensitive to terms of the order $1/m_b^3$ which is the highest order in the $1/m_b$ expansion included in the fit. The size of this terms together with the size of the α_s^2 corrections may serve as a conservative estimate of the uncertainties of the standard model calculation, which at the end determines the sensitivity to a possible new-physics contribution. Furthermore, an inclusion of higher moments in the fit, including the new contributions, (in particular with $i > 2$) needs the calculation of the $1/m_b^3$ terms for the new-physics contributions. As the impact of such hadronic mass moments to the fit is small we did not include a table of them.

			n	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
no E_l cuts	Tree-level	parton	0	1.0000	−0.6685	0.2212	0.5400	0.3315	−0.6597
			1	0.3072	−0.2092	0.0613	0.1372	0.0977	−0.2307
			2	0.1030	−0.0708	0.0188	0.0388	0.0314	−0.0845
			3	0.0365	−0.0252	0.0062	0.0118	0.0107	−0.0319
		μ_π^2/m_b^2 coeff.	0	−0.5000	0.3342	−0.0017	0.1703	−0.1652	0.3288
			1	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
			2	0.0858	−0.0590	0.0365	0.1146	0.0261	−0.0702
			3	0.0730	−0.0503	0.0210	0.0575	0.0214	−0.0637
		μ_g^2/m_b^2 coeff.	0	−1.9449	4.9934	1.0232	1.5624	−2.1536	3.7106
			1	−0.9625	1.8578	0.3253	0.6011	−0.7986	1.5873
			2	−0.4495	0.7237	0.1124	0.2427	−0.3081	0.6840
			3	−0.2052	0.2902	0.0410	0.1008	−0.1220	0.2966
	α_s/π coeff.		0	0.3125	0.8009	−2.6592	−8.8212	−2.1497	4.3637
			1	0.0908	0.2284	−0.7171	−2.3141	−0.5594	1.4880
			2	0.0276	0.0739	−0.2174	−0.6843	−0.1660	0.5394
			3	0.0085	0.0260	−0.0711	−0.2189	−0.0538	0.2039
$E_l > 1\,\mathrm{GeV}$ cut	Tree-level	parton	0	0.8148	−0.5617	0.1621	0.3586	0.2631	−0.6161
			1	0.2776	−0.1919	0.0520	0.1089	0.0867	−0.2232
			2	0.0979	−0.0678	0.0172	0.0340	0.0296	−0.0831
			3	0.0356	−0.0246	0.0059	0.0109	0.0104	−0.0317
		μ_π^2/m_b^2 coeff.	0	−0.4504	0.3225	0.0433	0.3440	−0.1479	0.3631
			1	0.0087	−0.0021	0.0564	0.2247	0.0031	0.0059
			2	0.0874	−0.0594	0.0377	0.1194	0.0267	−0.0691
			3	0.0733	−0.0504	0.0213	0.0583	0.0215	−0.0635
		μ_g^2/m_b^2 coeff.	0	−2.1029	4.6903	0.8592	1.4595	−2.0451	3.7102
			1	−0.9883	1.8078	0.2989	0.5845	−0.7805	1.5871
			2	−0.4540	0.7149	0.1078	0.2398	−0.3049	0.6840
			3	−0.2060	0.2886	0.0401	0.1003	−0.1214	0.2966
	α_s/π coeff.		0	0.2640	0.5740	−1.8506	−5.9374	−1.3992	3.9213
			1	0.0828	0.1930	−0.5920	−1.8692	−0.4440	1.4126
			2	0.0262	0.0679	−0.1964	−0.6098	−0.1467	0.5260
			3	0.0083	0.0249	−0.0674	−0.2058	−0.0504	0.2014

Table 5.1: *Tree level and α_s/π coefficients of the leptonic moments without E_l cuts and with a cut $E_l > 1\,\mathrm{GeV}$. Note that we have redefined $\mathbf{d_{L/R}} = m_B\, d_{L/R}$ and $\mathbf{g_{L/R}} = m_B\, g_{L/R}$ with $m_B = 5.279\,\mathrm{GeV}$ in order to tabulate dimensionless quantities.*

		i	j	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
Tree-level	parton	0	0	0.8148	−0.5617	0.1621	0.3586	0.2631	−0.6161
		0	1	0.3341	−0.2037	0.0682	0.1676	0.0922	−0.2365
		0	2	0.1411	−0.0761	0.0295	0.0789	0.0332	−0.0933
		0	3	0.0612	−0.0293	0.0131	0.0375	0.0123	−0.0378
		$i>0$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	μ_π^2/m_b^2 coeff.	0	0	−0.4504	0.3225	0.0433	0.3440	−0.1479	0.3631
		0	1	−0.4505	0.2921	−0.0597	−0.0843	−0.1329	0.3332
		0	2	−0.2673	0.1561	−0.0532	−0.1300	−0.0695	0.1841
		0	3	−0.1337	0.0706	−0.0327	−0.0935	−0.0308	0.0859
		1	0	−0.5424	0.3590	−0.1687	−0.4845	−0.1685	0.3887
		1	1	−0.1639	0.1022	−0.0598	−0.1852	−0.0478	0.1115
		1	2	−0.0417	0.0262	−0.0204	−0.0678	−0.0126	0.0273
		2	0	0.1203	−0.0547	0.0258	0.0742	0.0223	−0.0729
		2	1	0.0538	−0.0221	0.0118	0.0355	0.0087	−0.0306
		3	0	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	μ_g^2/m_b^2 coeff.	0	0	−2.1029	4.6903	0.8592	1.4595	−2.0451	3.7102
		0	1	−0.4609	1.2205	0.3461	0.4476	−0.5005	0.9855
		0	2	−0.0660	0.2921	0.1348	0.1332	−0.1119	0.2391
		0	3	0.0131	0.0538	0.0507	0.0363	−0.0194	0.0439
		1	0	0.3074	−0.5095	−0.0803	−0.1804	0.1654	−0.4093
		1	1	0.1171	−0.1971	−0.0381	−0.0751	0.0583	−0.1590
		1	2	0.0458	−0.0789	−0.0180	−0.0321	0.0211	−0.0635
		$i>1$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
α_s/π coeff.		0	0	0.2642	0.5739	−1.8506	−5.9373	−1.3992	3.9213
		0	1	0.1216	0.2462	−0.8449	−2.7806	−0.5529	1.6572
		0	2	0.0608	0.1057	−0.3919	−1.3149	−0.2221	0.7103
		0	3	0.0323	0.0455	−0.1842	−0.6272	−0.0907	0.3086
		1	0	0.0576	−0.0231	0.0018	0.0101	0.0018	−0.0079
		1	1	0.0288	−0.0108	0.0009	0.0052	0.0008	−0.0038
		1	2	0.0147	−0.0052	0.0004	0.0027	0.0004	−0.0018
		2	0	0.0046	−0.0016	0.0001	0.0007	0.0001	−0.0006
		2	1	0.0026	−0.0009	0.0000	0.0004	0.0000	−0.0003
		3	0	0.0007	−0.0002	0.0000	0.0001	0.0000	−0.0001

Table 5.2: *Tree level and α_s/π coefficients of the hadronic moments with a cut $E_l > 1\,\mathrm{GeV}$. The partonic tree-level moments for $i > 1$ are all zero. Note that we have redefined $\mathbf{d_{L/R}} = m_B\, d_{L/R}$ and $\mathbf{g_{L/R}} = m_B\, g_{L/R}$ with $m_B = 5.279\,\mathrm{GeV}$ in order to tabulate dimensionless quantities.*

		i	j	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
Tree-level	parton	0	0	1.0000	−0.6685	0.2212	0.5400	0.3315	−0.6597
		0	1	0.4220	−0.2500	0.0961	0.2556	0.1217	−0.2559
		0	2	0.1832	−0.0964	0.0429	0.1219	0.0461	−0.1021
		0	3	0.0815	−0.0383	0.0196	0.0586	0.0180	−0.0418
		$i>0$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
	μ_π^2/m_b^2 coeff.	0	0	−0.5000	0.3342	−0.0017	0.1703	−0.1652	0.3288
		0	1	−0.5000	0.3342	−0.100	−0.2229	−0.1652	0.3288
		0	2	−0.2902	0.1836	−0.0773	−0.2119	−0.0899	0.1840
		0	3	−0.1382	0.0837	−0.0448	−0.1348	−0.0406	0.0853
		1	0	−0.5780	0.4185	−0.2038	−0.5937	−0.2091	0.4025
		1	1	−0.1584	0.1172	−0.0695	−0.2158	−0.0585	0.1129
		1	2	−0.0283	0.0280	−0.0217	−0.0718	−0.0143	0.0258
		2	0	0.1609	−0.0728	0.0386	0.1159	0.0337	−0.0809
		2	1	0.0735	−0.0302	0.0180	0.0561	0.0138	−0.0343
		3	0	0.0000	0.0000	0.0000	0.0000	0.0000	−0.0000
	μ_g^2/m_b^2 coeff.	0	0	−1.9449	4.9934	1.0232	1.5624	−2.1536	3.7106
		0	1	−0.3850	1.2777	0.4097	0.4782	−0.5223	0.9700
		0	2	−0.0302	0.2833	0.1576	0.1391	−0.1109	0.2254
		0	3	0.0298	0.0342	0.0578	0.0350	−0.0146	0.0347
		1	0	0.3143	−0.6395	−0.1100	−0.2167	0.2027	−0.4360
		1	1	0.1195	−0.2561	−0.0529	−0.0925	0.0744	−0.1709
		1	2	0.0466	−0.1059	−0.0254	−0.0405	0.0282	−0.0689
		$i>1$	j	0.0000	0.0000	0.0000	0.0000	0.0000	0.0000
α_s/π coeff.		0	0	0.3128	0.8007	−2.6592	−8.8212	−2.1497	4.3637
		0	1	0.1631	0.3441	−1.2391	−4.1901	−0.8839	1.8575
		0	2	0.0910	0.1477	−0.5850	−2.0067	−0.3694	0.8017
		0	3	0.0526	0.0632	−0.2793	−0.9681	−0.1568	0.3505
		1	0	0.0901	−0.0363	0.0028	0.0176	0.0032	−0.0095
		1	1	0.0470	−0.0178	0.0014	0.0093	0.0015	−0.0046
		1	2	0.0251	−0.0090	0.0007	0.0050	0.0007	−0.0023
		2	0	0.0091	−0.0033	0.0001	0.0015	0.0002	−0.0008
		2	1	0.0053	−0.0019	0.0000	0.0009	0.0001	−0.0004
		3	0	0.0018	−0.0006	0.0000	0.0003	0.0000	−0.0001

Table 5.3: *Tree level and α_s/π coefficients of the hadronic moments without E_l cuts. The partonic tree-level moments for $i > 1$ are all zero. Note that we have redefined $\mathbf{d_{L/R}} = m_B\, d_{L/R}$ and $\mathbf{g_{L/R}} = m_B\, g_{L/R}$ with $m_B = 5.279\,\mathrm{GeV}$ in order to tabulate dimensionless quantities.*

μ	n	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
2.3 GeV	0	1.0253	−0.6037	0.0042	−0.1916	0.1533	−0.2983
	1	0.3145	−0.1907	0.0028	−0.0552	0.0512	−0.1074
	2	0.1052	−0.0648	0.0011	−0.0182	0.0176	−0.0397
	3	0.0372	−0.0231	0.0004	−0.0065	0.0062	−0.0150
4.6 GeV	0	1.0208	−0.6151	0.0441	−0.0474	0.1883	−0.3692
	1	0.3132	−0.1940	0.0135	−0.0169	0.0604	−0.1317
	2	0.1048	−0.0658	0.0043	−0.0068	0.0204	−0.0485
	3	0.0371	−0.0234	0.0015	−0.0028	0.0071	−0.0184
9.2 GeV	0	1.0177	−0.6231	0.0715	0.0752	0.2146	−0.4223
	1	0.3123	−0.1963	0.0208	0.0164	0.0674	−0.1499
	2	0.1046	−0.0666	0.0065	0.0034	0.0225	−0.0552
	3	0.0370	−0.0237	0.0022	0.0005	0.0078	−0.0209

Table 5.4: *Summed up tree level and α_s/π coefficients of the leptonic moments without E_l cuts for $\mu = 2.3, 4.6$ and $9.2\,\mathrm{GeV}$.*

μ	i	j	c_L^2	$c_L c_R$	$c_L g_L$	$c_L g_R$	$c_L d_L$	$c_L d_R$
2.3 GeV	0	0	1.0253	−0.6037	0.0042	−0.1916	0.1533	−0.2983
	0	1	0.4352	−0.2222	−0.0051	−0.0914	0.0486	−0.1024
	0	2	0.1906	−0.0845	−0.0049	−0.0440	0.0156	−0.0360
	0	3	0.0857	−0.0331	−0.0033	−0.0214	0.0050	−0.0129
4.6 GeV	0	0	1.0208	−0.6151	0.0441	−0.0474	0.1883	−0.3692
	0	1	0.4329	−0.2271	0.0136	−0.0234	0.0628	−0.1322
	0	2	0.1892	−0.0866	0.0040	−0.0117	0.0215	−0.0487
	0	3	0.0850	−0.0340	0.0010	−0.0059	0.0075	−0.0185
9.2 GeV	0	0	1.0177	−0.6231	0.0715	0.0752	0.2146	−0.4223
	0	1	0.4312	−0.2305	0.0269	0.0335	0.0734	−0.1545
	0	2	0.1883	−0.0880	0.0104	0.0151	0.0258	−0.0582
	0	3	0.0845	−0.0347	0.0041	0.0069	0.0093	−0.0226

Table 5.5: *Summed up tree level and α_s/π coefficients of the non-zero-tree-level hadronic moments without E_l cuts for $\mu = 2.3, 4.6$ and $9.2\,\mathrm{GeV}$.*

6 Limit on a Right-Handed Admixture to the Weak $b \to c$ Current from Inclusive Semileptonic Decays

6.1 Transistion from Theory

As we have argued in 5.6 that the lepton-energy moments and hadronic-mass moments are not sensitive to the parameters g_{L}, g_{R}, d_{L} and d_{R}. Thus we will perform the anticipated combined fit of the moments only including possible right-handed contributions to the $b \to c$ transition by analysing the parameter c_{R}. The moments are calculated by squaring the matrix element, leading to the occurence of the parameters in pairs, with the leading contributions being c_{L}^2 and $c_{\mathrm{L}} c_{\mathrm{R}}$ as used throughout our calculations in the previous chapter. The parameter c_{L} is an overall factor like $|V_{cb}|$ and has to be absorbed into the latter. Otherwise c_{L} and $|V_{cb}|$ were 100 % correlated giving no additional information. The parameter used in the fit is therefore $c'_{\mathrm{R}} = c_{\mathrm{R}}/c_{\mathrm{L}}$.

We will perform the analysis in collaboration with Verena Klose an author of [34], Heiko Lacker and Thomas Lück by using their HQEfitter program. The theory predictions for the fit in [34] are calculated by `FORTAN` routines developed by Paolo Gambino, Nikolai Uraltsev and Paolo Giordano [18, 30]. The setup of calculation differs slightly from ours, due to meet the requirements of a fit. The normalizations are thus chosen differently, the change from the pole mass scheme to the kinetic scheme is handled slightly differently. The functions of radiative corrections depending on the lower electron energy cut and the mass ratio $\rho = m_{\mathrm{c}}^2/m_{\mathrm{b}}^2$ are fitted polynoms in the depending variables to provide a fast evaluation in the combined HQE fit. The fit with the inclusion of possible right-handed contributions will be performed by extending the “standard model” fit as descibed in [34]. The changes of our calculations to match the `FORTAN` routines are descibed in section 6.2.3.

In section 6.2.2 we will explain in detail the extraction of the CKM matrix element $|V_{cb}|$ in the fit. This extraction is crucial in the standard model HQE fit, because the precise theoretical desciption and experimental determinations of the inclusive decay $\overline{B} \to X_c \ell \overline{\nu}_\ell$ allows to extract a contrained value of $|V_{cb}|$. The value of $|V_{cb}|$ from inclusive decays, i. e. the HQE fit, has a relative precision of about 2 %, while the competitive determination from exclusive decays about 3 %. Thus inclusive B decays give the most precise value of the determination of $|V_{cb}|$ entering the CKM fit. On the other hand the value of $|V_{cb}|$ from inclusive decays is not in agreement with the value from exclusive decays. The hypothesis of a right-handed contribution to the $b \to c$ current might provide a solution to this tension by tracing the tension back to effects from new physics contributions, which we will investigate in section 6.4.

6.2 Analysis

6.2.1 Fit Setup

The combined fit for the extraction of the new parameter c'_{R} is performed along the lines as described in [34]. It is based on the χ^2 minimization,

$$\chi^2 = \left(\overrightarrow{M}_{\mathrm{exp}} - \overrightarrow{M}_{\mathrm{theo}}\right)^{\mathrm{T}} \mathcal{C}_{\mathrm{tot}}^{-1} \left(\overrightarrow{M}_{\mathrm{exp}} - \overrightarrow{M}_{\mathrm{theo}}\right), \tag{6.1}$$

with the included measured moments $\overrightarrow{M}_{\mathrm{exp}}$, the corresponding theoretical prediction of these moments $\overrightarrow{M}_{\mathrm{theo}}$ and the total covariance matrix $\mathcal{C}_{\mathrm{tot}}$ defined as the sum of the experimental ($\mathcal{C}_{\mathrm{exp}}$) and the theoretical ($\mathcal{C}_{\mathrm{theo}}$) covariance matrix, respectively.

In the analysis of [34] the theoretical prediction for the moments $\overrightarrow{M}_{\mathrm{HQE}}$ are calculated perturbatively in a heavy-quark expansion (HQE) in the kinetic-mass scheme up to $\mathcal{O}(1/m_b^3)$ with perturbative contributions [18, 30, 35] resulting in a dependence on six parameters: the running masses of the b- and c-quarks, $m_b(\mu)$ and $m_c(\mu)$, the parameters μ_π^2 and μ_G^2 at $\mathcal{O}(1/m_b^2)$ in the HQE, and, at $\mathcal{O}(1/m_b^3)$, the parameters ρ_D^3 and ρ_{LS}^3.

New in this analysis is the inclusion of possible right-handed quark currents in the calculation of the theoretical prediction of the moments. The right-handed contributions are calculated and used here up to $\mathcal{O}(1/m_b^2)$ in the HQE and $\mathcal{O}(\alpha_s)$ in the perturbative correction. The aim of this fit is to give an upper bound for the relative contribution of a right-handed current compared with the standard-model left-handed current, which is parametrized by a prefactor c'_{R} for the new contributions to test. Thus the theoretical predition of the moments depends on seven parameters to fit:

$$\overrightarrow{M}_{\mathrm{theo}} = \overrightarrow{M}_{\mathrm{theo}}(c'_{\mathrm{R}}, m_b, m_c, \mu_\pi^2, \mu_G^2, \rho_D^3, \rho_{LS}^3).$$

6.2.2 Determination of $|V_{cb}|$

In the presence of a right-handed mixture the definition of the parameter $|V_{cb}|$ becomes ambiguous. Out of the three parameters $|V_{cb}|$, c_{L}, c'_{R} only two are independent, since c_{L} can be absorbed into $|V_{cb}|$. To this end we choose to define

$$|V_{cb}|\bar{b}_L\gamma_\mu c_L \quad \rightarrow \quad |V_{cb}|\left(\bar{b}_L\gamma_\mu c_L + c'_{\mathrm{R}}\bar{b}_R\gamma_\mu c_R\right).$$

The determination of the $|V_{cb}|$ in the fit package uses a linearized form of the semileptonic $b \rightarrow c$ rate $\Gamma_{cl\nu}$ from [35], which we extend by the result for the right-handed contribution:

$$\begin{aligned}\Gamma_{cl\nu} = & \frac{\mathrm{G_F^2}|V_{cb}|^2 m_{\mathrm{b}}^5}{192\pi^3}\left\{(1+A_{\mathrm{ew}})A^{\mathrm{pert}}\left[z_0(\rho)\left(1-\frac{\mu_\pi^2-\mu_G^2+\frac{\rho_D^3+\rho_{LS}^3}{m_{\mathrm{b}}}}{2m_{\mathrm{b}}^2}\right)\right.\right. \\ & \left.\left. -2(1-\rho)^4\frac{\mu_G^2-\frac{\rho_D^3+\rho_{LS}^3}{m_{\mathrm{b}}}}{m_{\mathrm{b}}^2}+d(\rho)\frac{\rho_D^3}{m_{\mathrm{b}}^3}\right]+c'_{\mathrm{R}}\left[z_0^{c'_{\mathrm{R}}}(\rho)\left(1-\frac{\mu_\pi^2}{2m_{\mathrm{b}}^2}\right)+z_{\mu_G^2}^{c'_{\mathrm{R}}}(\rho)\frac{\mu_G^2}{m_{\mathrm{b}}^2}\right]\right\}.\end{aligned} \tag{6.2}$$

where $\rho = m_{\mathrm{c}}^2/m_{\mathrm{b}}^2$ and A_{ew} is the electroweak correction coresponding to the renormalization of the Fermi interaction ($1 + A_{\mathrm{ew}} \approx 1.014$) and A^{pert} accounts for the perturbative corrections up to second order and all orders in the BLM corrections ($A^{\text{pert}} \approx 0.908$ for $m_{\text{c}}/m_{\text{b}} = 0.25$ and the scale at $\mu = 1\,\text{GeV}$). The quantity z_0 is the tree-level phase space factor:

$$z_0(\rho) = 1 - 8\rho + 8\rho^3 - \rho^4 - 12\rho^2 \ln \rho \tag{6.3}$$

and $z_0^{c'_{\text{R}}}(\rho)$ is the corresponding phase space factor for the right-handed contribution (see (4.51)):

$$z_0^{c'_{\text{R}}}(\rho) = -4\sqrt{\rho}\left(1 + 9\rho - 9\rho^2 - \rho^3 + 6\rho(1+\rho)\log\rho\right) \tag{6.4}$$

and $z_{\mu_G^2}^{c'_{\text{R}}}(\rho)$ the normalized coefficient of the right-handed μ_G^2 correction (see also (4.51)):

$$z_{\mu_G^2}^{c'_{\text{R}}}(\rho) = -\frac{2}{3}\sqrt{\rho}\left(13 - 27\rho + 27\rho^2 - 13\rho^2 + 6(3\rho^2 - 3\rho + 2)\log\rho\right). \tag{6.5}$$

The semileptonic $b \to c$ rate (6.2) can first be related to the semileptonic $b \to c$ branching fraction $\mathcal{B}_{clv} = \Gamma_{cl\nu}/\Gamma$ and the B-meson lifetime $\tau_B = 1/\Gamma$ via

$$\Gamma_{cl\nu} = \frac{\mathcal{B}_{clv}}{\tau_B}. \tag{6.6}$$

The semileptonic $b \to c$ branching fraction $\mathcal{B}_{clv}$ is calculated from the total semileptonic branching fraction $\mathcal{B}_{SL}$ subtracting the $b \to u$ branching fraction $\mathcal{B}_{ulv}$, i. e.

$$\mathcal{B}_{clv} = \mathcal{B}_{SL} - \mathcal{B}_{ulv} = 0.105 - 0.0018 = 0.1032.$$

Factoring out $|V_{cb}|^2$ from the semileptonic $b \to c$ rate ($|V_{cb}|^2(\Gamma_{cl\nu}/|V_{cb}|^2)$) allows us to find an expression for the determination of $|V_{cb}|$:

$$|V_{cb}| = \sqrt{\frac{\mathcal{B}_{clv}}{\tau_B}}\frac{1}{\sqrt{\frac{\Gamma_{cl\nu}}{|V_{cb}|^2}}} \tag{6.7}$$

In order to obtain small expansion coefficients in a linearized form (6.7) is normalized to $V_{\text{cb},0}$=0.0417, $\mathcal{B}_{clv,0}$=0.1032, $\tau_{B,0}$=1.55 ps and $(\Gamma_{cl\nu}/|V_{cb}|^2)_0$=$2.52431 \cdot 10^{-11}$ with an a-priory estimates of the parameters as follows

$$\begin{array}{llll} m_{\text{b}} = 4.60\,\text{GeV} & \mu_\pi^2 = 0.40\,\text{GeV} & \rho_D^3 = 0.20\,\text{GeV} & \alpha_s = 0.22 \\ m_{\text{c}} = 1.15\,\text{GeV} & \mu_G^2 = 0.35\,\text{GeV} & \rho_{LS}^3 = -0.15\,\text{GeV} & \rho = m_{\text{c}}^2/m_{\text{b}}^2 = 1/16 \end{array} \tag{6.8}$$

yielding

$$\frac{|V_{cb}|}{0.0417} = \sqrt{\frac{\mathcal{B}_{clv}}{0.1032}\frac{1.55}{\tau_B}}\sqrt{\frac{2.52431 \cdot 10^{-11}}{\Gamma_{cl\nu}/|V_{cb}|^2}}. \tag{6.9}$$

To obtain the linear expression for $|V_{cb}|$ the expression $\sqrt{\frac{2.52431 \cdot 10^{-11}}{(\Gamma_{cl\nu}/|V_{cb}|^2)}}$ in (6.9) is expanded in a Taylor series in the parameters listed in (6.8) to first order each around their a-priory estimate. Using the numerical values of the normalization terms the result reads

$$\begin{aligned} \frac{|V_{cb}|}{0.0417} = & \sqrt{\frac{\mathcal{B}_{clv}}{0.1032}\frac{1.55}{\tau_B}}\left[1 + 0.30\,(\alpha_s(m_{\text{b}}) - 0.22)\right] \\ & \times \left[1 - 0.66\,(m_{\text{b}} - 4.60) + 0.39\,(m_{\text{c}} - 1.15)\right. \\ & \quad + 0.013\,(\mu_\pi^2 - 0.40) + 0.09\,(\rho_D^3 - 0.20) \\ & \quad + 0.05\,(\mu_G^2 - 0.35) - 0.01\,(\rho_{LS}^3 + 0.15) \\ & \quad \left. + 0.341\,c'_{\text{R}}\right]. \end{aligned} \tag{6.10}$$

Note the last term ($0.341\, c'_{\mathrm{R}}$), taking into account the possible contributions from a right-handed quark current. The a-priory estimate of c'_{R} is zero, i. e. the standard-model value. Due to the sizable factor, a positiv c'_{R} increases $|V_{cb}|$ compared to the standard-model fit without c'_{R}.

The total branching fraction $\mathcal{B}(\overline{B} \to X_c \ell^- \overline{\nu})$ in the fit is extrapolated from measured partial branching fractions $\mathcal{B}_{p^*_{\ell,\min}}(\overline{B} \to X_c \ell^- \overline{\nu})$, with $p^*_\ell \geq p^*_{\ell,\min}$. This is done by comparison with the HQE prediction of the relative decay fraction (r.h.s):

$$\frac{\mathcal{B}_{p^*_{\ell,\min}}(\overline{B} \to X_c \ell^- \overline{\nu})}{\mathcal{B}(\overline{B} \to X_c \ell^- \overline{\nu})} = \frac{\int_{p^*_{\ell,\min}} \frac{\mathrm{d}\Gamma_{SL}}{\mathrm{d}E_{\mathrm{l}}} \mathrm{d}E_{\mathrm{l}}}{\int_0 \frac{\mathrm{d}\Gamma_{SL}}{\mathrm{d}E_{\mathrm{l}}} \mathrm{d}E_{\mathrm{l}}}. \tag{6.11}$$

Thus the total branching fraction can be introduced as a free parameter in the fit. By adding the average B-meson lifetime τ_B to the measured and predicted values, $|V_{cb}|$ can as well be introduced as a free parameter using (6.10).

6.2.3 Calculation of the Theory Prediction

For the combined fit we are about to perform we assemble our calculations of the non-perturbative and perturbative corrections of the decay at hand. Even though we computed observables aiming at such an evaluation, we should discuss the exact definitions used in the combined fit, because the theory prediction in the adapted standard model fit are computed in `FORTRAN 77` code originally developed mainly by Paolo Gambino, Nicolai Uraltsev and Paolo Giordano [18, 30].

Prediction of the Relative Branching Fraction

The HQE prediction of the relative branching fraction is calculated directly in its expansion in α_s and $1/m_{\mathrm{b}}$. In our earlier calculations we normalized the various corrections to the tree and parton level result of the total rate. In the combined fit the theoretical predictions for the total rate and the moments is to be compared with the experimental measurements, or more precisely the fit parameters are adjusted to a set of values minimizing the sum of the squared differences of the predicted and measured observables, i. e. the Least Squares Method. The experimental values for the moments can only be normalized to the measured total rate, which includes in a way all corrections, because you can't switch them off in the detector. As we have to account for that, the theoretical predictions have to be normalized to the total rate including all known corrections or at least the order of corrections used in the fit to state the best estimation of the real value. As discussed in (6.11) the total branching fraction is extrapolated with the help of the relative decay fraction $R^*(\xi)$ which is the ratio of the total rate with an electron energy cut $\xi = \frac{2E_{\mathrm{cut}}}{m_{\mathrm{b}}}$ and the total rate without a cut. Such combinations are subjected to strong cancellations especially in the perturbative correction. For best numerical results the following combinations of coefficients is needed to perform:

$$R^*(\xi) = \frac{\int_{p^*_{\ell,\min}} \frac{\mathrm{d}\Gamma_{SL}}{\mathrm{d}E_{\mathrm{l}}} \mathrm{d}E_{\mathrm{l}}}{\int_0 \frac{\mathrm{d}\Gamma_{SL}}{\mathrm{d}E_{\mathrm{l}}} \mathrm{d}E_{\mathrm{l}}} = \frac{\Gamma(\xi)}{\Gamma(0)} \tag{6.12}$$

$$= \frac{\Gamma^{(0,0)}(\xi) + \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}} \delta m_{\mathrm{b}}^{\mathrm{kin}} + \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}} \delta m_{\mathrm{c}}^{\mathrm{kin}} + \frac{\alpha_s}{\pi} \Gamma^{(0,1)}(\xi) + \frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2} \Gamma^{(2,0)}_{\hat{\mu}_\pi^2}(\xi) + \frac{\hat{\mu}_{\mathrm{G}}^2}{m_{\mathrm{b}}^2} \Gamma^{(2,0)}_{\hat{\mu}_{\mathrm{G}}^2}(\xi) + \dots}{\Gamma^{(0,0)}(0) + \frac{\mathrm{d}\Gamma^{(0,0)}(0)}{\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}} \delta m_{\mathrm{b}}^{\mathrm{kin}} + \frac{\mathrm{d}\Gamma^{(0,0)}(0)}{\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}} \delta m_{\mathrm{c}}^{\mathrm{kin}} + \frac{\alpha_s}{\pi} \Gamma^{(0,1)}(0) + \frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2} \Gamma^{(2,0)}_{\hat{\mu}_\pi^2}(0) + \frac{\hat{\mu}_{\mathrm{G}}^2}{m_{\mathrm{b}}^2} \Gamma^{(2,0)}_{\hat{\mu}_{\mathrm{G}}^2}(0) + \dots} \tag{6.13}$$

$$= \frac{\Gamma^{(0,0)}(\xi)}{\Gamma^{(0,0)}(0)}\Bigg[1 + \delta m_{\mathrm{b}}^{\mathrm{kin}}\left(\frac{\mathrm{d}\Gamma^{(0,0)}(\xi)/\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}}{\Gamma^{(0,0)}(\xi)} - \frac{\mathrm{d}\Gamma^{(0,0)}(0)/\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}}{\Gamma^{(0,0)}(0)}\right) \tag{6.14}$$

$$+ \delta m_{\mathrm{c}}^{\mathrm{kin}}\left(\frac{\mathrm{d}\Gamma^{(0,0)}(\xi)/\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}}{\Gamma^{(0,0)}(\xi)} - \frac{\mathrm{d}\Gamma^{(0,0)}(0)/\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}}{\Gamma^{(0,0)}(0)}\right)$$

$$+ \frac{\alpha_s}{\pi}\left(\frac{\Gamma^{(0,1)}(\xi)}{\Gamma^{(0,0)}(\xi)} - \frac{\Gamma^{(0,1)}(0)}{\Gamma^{(0,0)}(0)}\right) + \frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2}\left(\frac{\Gamma^{(2,0)}_{\hat{\mu}_\pi^2}(\xi)}{\Gamma^{(0,0)}(\xi)} - \frac{\Gamma^{(2,0)}_{\hat{\mu}_\pi^2}(0)}{\Gamma^{(0,0)}(0)}\right)$$

$$+ \frac{\hat{\mu}_{\mathrm{G}}^2}{m_{\mathrm{b}}^2}\left(\frac{\Gamma^{(2,0)}_{\hat{\mu}_{\mathrm{G}}^2}(\xi)}{\Gamma^{(0,0)}(\xi)} - \frac{\Gamma^{(2,0)}_{\hat{\mu}_{\mathrm{G}}^2}(0)}{\Gamma^{(0,0)}(0)}\right) + \ldots\Bigg]$$

$$= \frac{\Gamma^{(0,0)}(\xi)}{\Gamma^{(0,0)}(0)}\left[1 + \delta R^*_{m_{\mathrm{b}}^{\mathrm{kin}}}(\xi) + \delta R^*_{m_{\mathrm{c}}^{\mathrm{kin}}}(\xi) + \frac{\alpha_s}{\pi}R^{*(0,1)}(\xi) + \frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2}R^{*(2,0)}_{\hat{\mu}_\pi^2}(\xi) + \frac{\hat{\mu}_{\mathrm{G}}^2}{m_{\mathrm{b}}^2}R^{*(2,0)}_{\hat{\mu}_{\mathrm{G}}^2}(\xi) + \ldots\right] \tag{6.15}$$

where $\Gamma^{(m,n)}(\xi)$ indicates the coefficient of the total rate at order m in the $1/m_{\mathrm{b}}$ expansion and at order n in the perturbative expansion. A subscripted $\hat{\mu}_\pi^2$ or $\hat{\mu}_{\mathrm{G}}^2$ indicates the exact coefficient of the corresponding HQE parameter. The various R^* expressions represent the differences appearing in the expansion in the second step. The ellipses stand for the terms from order $1/m_{\mathrm{b}}^3$ and the BLM corrections, which are used in the standard model part of the fit, but not for the right handed admixture. The terms $\frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}}\delta m_{\mathrm{b}}^{\mathrm{kin}}$ and $\frac{\mathrm{d}\Gamma^{(0,0)}(0)}{\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}}\delta m_{\mathrm{c}}^{\mathrm{kin}}$ account for the change of mass scheme from the pole scheme to the kinetic scheme, as already discussed in 5.5.1. Due to the different normalization at this evaluation the change of scheme is implemented as a taylor expansion of the lowest order in the bottom and charm quark *kinetic* masses with the same expression as (2.80) for the bottom quark, not showing the BLM corrections to the masses for the standard model part, which is used in the fit:

$$\delta m_{\mathrm{b}}^{\mathrm{kin}}(\mu_f) = \frac{\alpha_s}{\pi}\left(\frac{16}{9}\mu_f + \frac{2\mu_f^2}{3m_{\mathrm{b}}}\right), \tag{6.16}$$

$$\delta m_{\mathrm{c}}^{\mathrm{kin}}(\mu_f) = \frac{\alpha_s}{\pi}\left(\frac{16}{9}\mu_f + \frac{2\mu_f^2}{3m_{\mathrm{c}}}\right). \tag{6.17}$$

The tree and parton level of the total rate depends on m_{b} throught $\xi = \frac{2E_{\mathrm{cut}}}{m_{\mathrm{b}}}$ and $\rho = m_{\mathrm{c}}^2/m_{\mathrm{b}}^2$ and on m_{c} only through $\rho = m_{\mathrm{c}}^2/m_{\mathrm{b}}^2$:

$$\frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}} = \frac{\partial\Gamma^{(0,0)}(\xi)}{\partial\xi}\frac{\partial\xi}{\partial m_{\mathrm{b}}} + \frac{\partial\Gamma^{(0,0)}(\xi)}{\partial\rho}\frac{\partial\rho}{\partial m_{\mathrm{b}}} = -\frac{\xi}{m_{\mathrm{b}}}\frac{\partial\Gamma^{(0,0)}(\xi)}{\partial\xi} - \frac{2\rho}{m_{\mathrm{b}}}\frac{\partial\Gamma^{(0,0)}(\xi)}{\partial\rho}, \tag{6.18}$$

$$\frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}} = \frac{\partial\Gamma^{(0,0)}(\xi)}{\partial\rho}\frac{\partial\rho}{\partial m_{\mathrm{c}}} = \frac{2\rho}{m_{\mathrm{c}}}\frac{\partial\Gamma^{(0,0)}(\xi)}{\partial\rho}. \tag{6.19}$$

Prediction of the Moments of the Lepton Energy Spectrum

As shown in section 4.5.4 we need to calculate the *charged lepton energy moments*:

Order	Moment
0	Γ
1	$\langle E_\mathrm{l} \rangle$
2	$\langle (E_\mathrm{l} - \langle E_\mathrm{l} \rangle)^2 \rangle$
3	$\langle (E_\mathrm{l} - \langle E_\mathrm{l} \rangle)^3 \rangle$

(6.20)

and as explained in 4.5.4 we can calculate $\langle E_\mathrm{l}^1 \rangle$, $\langle E_\mathrm{l}^2 \rangle$ and $\langle E_\mathrm{l}^3 \rangle$ instead. The fitter uses $y = \frac{2}{m_\mathrm{b}} E_\mathrm{l}$ again. Thus we find expressions for the charged lepton energy moments normalized to the total decay rate with cut $\Gamma(\xi)$ (in section 4.5.4 we normalized only to the tree and parton level of the total decay rate):

$$L_n(\xi) = \frac{\langle E_\mathrm{l}^n \rangle(\xi)}{\Gamma(\xi)} = \left(\frac{m_\mathrm{b}}{2}\right)^n \frac{\langle y^n \rangle(\xi)}{\Gamma(\xi)} \tag{6.21}$$

$$= \left(\frac{m_\mathrm{b}}{2}\right)^n \frac{\langle y^n \rangle^{(0,0)}(\xi) + \frac{\mathrm{d}\langle y^n \rangle^{(0,0)}(\xi)}{\mathrm{d}m_\mathrm{b}^\mathrm{kin}} \delta m_\mathrm{b}^\mathrm{kin} + \frac{\mathrm{d}\langle y^n \rangle^{(0,0)}(\xi)}{\mathrm{d}m_\mathrm{c}^\mathrm{kin}} \delta m_\mathrm{c}^\mathrm{kin}}{\Gamma^{(0,0)}(\xi) + \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_\mathrm{b}^\mathrm{kin}} \delta m_\mathrm{b}^\mathrm{kin} + \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_\mathrm{c}^\mathrm{kin}} \delta m_\mathrm{c}^\mathrm{kin} + \frac{\alpha_s}{\pi}\Gamma^{(0,1)}(\xi) + \frac{\hat\mu_\pi^2}{m_\mathrm{b}^2}\Gamma^{(2,0)}_{\hat\mu_\pi^2}(\xi) + \frac{\hat\mu_\mathrm{G}^2}{m_\mathrm{b}^2}\Gamma^{(2,0)}_{\hat\mu_\mathrm{G}^2}(\xi) + \ldots} \tag{6.22}$$

$$+ \left(\frac{m_\mathrm{b}}{2}\right)^n \frac{\frac{\alpha_s}{\pi} \langle y^n \rangle^{(0,1)}(\xi) + \frac{\hat\mu_\pi^2}{m_\mathrm{b}^2} \langle y^n \rangle^{(2,0)}_{\hat\mu_\pi^2}(\xi) + \frac{\hat\mu_\mathrm{G}^2}{m_\mathrm{b}^2} \langle y^n \rangle^{(2,0)}_{\hat\mu_\mathrm{G}^2}(\xi) + \ldots}{\Gamma^{(0,0)}(\xi) + \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_\mathrm{b}^\mathrm{kin}} \delta m_\mathrm{b}^\mathrm{kin} + \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)}{\mathrm{d}m_\mathrm{c}^\mathrm{kin}} \delta m_\mathrm{c}^\mathrm{kin} + \frac{\alpha_s}{\pi}\Gamma^{(0,1)}(\xi) + \frac{\hat\mu_\pi^2}{m_\mathrm{b}^2}\Gamma^{(2,0)}_{\hat\mu_\pi^2}(\xi) + \frac{\hat\mu_\mathrm{G}^2}{m_\mathrm{b}^2}\Gamma^{(2,0)}_{\hat\mu_\mathrm{G}^2}(\xi) + \ldots} \tag{6.23}$$

$$\begin{aligned} = \left(\frac{m_\mathrm{b}}{2}\right)^n \frac{\langle y^n \rangle^{(0,0)}(\xi)}{\Gamma^{(0,0)}(\xi)} \Bigg[1 &+ \delta m_\mathrm{b}^\mathrm{kin} \left(\frac{\mathrm{d}\langle y^n \rangle^{(0,0)}(\xi)/\mathrm{d}m_\mathrm{b}^\mathrm{kin}}{\langle y^n \rangle^{(0,0)}(\xi)} - \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)/\mathrm{d}m_\mathrm{b}^\mathrm{kin}}{\Gamma^{(0,0)}(\xi)} \right) \\ &+ \delta m_\mathrm{c}^\mathrm{kin} \left(\frac{\mathrm{d}\langle y^n \rangle^{(0,0)}(\xi)/\mathrm{d}m_\mathrm{c}^\mathrm{kin}}{\langle y^n \rangle^{(0,0)}(\xi)} - \frac{\mathrm{d}\Gamma^{(0,0)}(\xi)/\mathrm{d}m_\mathrm{c}^\mathrm{kin}}{\Gamma^{(0,0)}(\xi)} \right) \\ &+ \frac{\alpha_s}{\pi} \left(\frac{\langle y^n \rangle^{(0,1)}(\xi)}{\langle y^n \rangle^{(0,0)}(\xi)} - \frac{\Gamma^{(0,1)}(\xi)}{\Gamma^{(0,0)}(\xi)} \right) + \frac{\hat\mu_\pi^2}{m_\mathrm{b}^2} \left(\frac{\langle y^n \rangle^{(2,0)}_{\hat\mu_\pi^2}(\xi)}{\langle y^n \rangle^{(0,0)}(\xi)} - \frac{\Gamma^{(2,0)}_{\hat\mu_\pi^2}(\xi)}{\Gamma^{(0,0)}(\xi)} \right) \\ &+ \frac{\hat\mu_\mathrm{G}^2}{m_\mathrm{b}^2} \left(\frac{\langle y^n \rangle^{(2,0)}_{\hat\mu_\mathrm{G}^2}(\xi)}{\langle y^n \rangle^{(0,0)}(\xi)} - \frac{\Gamma^{(2,0)}_{\hat\mu_\mathrm{G}^2}(\xi)}{\Gamma^{(0,0)}(\xi)} \right) + \ldots \Bigg] \end{aligned} \tag{6.24}$$

$$\begin{aligned} = \left(\frac{m_\mathrm{b}}{2}\right)^n \frac{\langle y^n \rangle^{(0,0)}(\xi)}{\Gamma^{(0,0)}(\xi)} \Bigg[1 &+ \delta L_{n,m_\mathrm{b}^\mathrm{kin}}(\xi) + \delta L_{n,m_\mathrm{c}^\mathrm{kin}}(\xi) + \frac{\alpha_s}{\pi} L_n^{(0,1)}(\xi) \\ &+ \frac{\hat\mu_\pi^2}{m_\mathrm{b}^2} L^{(2,0)}_{n,\hat\mu_\pi^2}(\xi) + \frac{\hat\mu_\mathrm{G}^2}{m_\mathrm{b}^2} L^{(2,0)}_{n,\hat\mu_\mathrm{G}^2}(\xi) + \ldots \Bigg]. \end{aligned} \tag{6.25}$$

Prediction of the Moments of the Hadronic Invariant Mass

The experimental moments of the hadronic invariant mass are normalized to the total decay rate, but for their theoretical prediction a normalization to the partonic rate is sufficient. Different from our earlier evaluation the normalizing partonic rate has to be computed with a cut on the electron energy, because the experimental total rate cannot be extracted without such a cut. Analogously to (4.63) we calulate

$$\left\langle m_X^2 \right\rangle = m_c^2 + \Lambda^2 + 2\Lambda m_b \langle E_0 \rangle + m_b^2 \langle \hat{\mathrm{s}}_0 - \rho \rangle \tag{6.26}$$

$$\begin{aligned} \left\langle \left(m_X^2 - \langle m_X^2 \rangle\right)^2 \right\rangle = {} & m_b^4 \left(\langle (\hat{\mathrm{s}}_0 - \rho)^2 \rangle - \langle \hat{\mathrm{s}}_0 - \rho \rangle^2 \right) \\ & + 4\Lambda m_b^3 \left(\langle \hat{E}_0 (\hat{\mathrm{s}}_0 - \rho) \rangle - \langle \hat{E}_0 \rangle \langle \hat{\mathrm{s}}_0 - \rho \rangle \right) \\ & + 4\Lambda^2 m_b^2 \left(\langle \hat{E}_0^2 \rangle - \langle \hat{E}_0 \rangle^2 \right) \end{aligned} \tag{6.27}$$

$$\begin{aligned} \left\langle \left(m_X^2 - \langle m_X^2 \rangle\right)^3 \right\rangle = {} & 12\Lambda^2 m_b^4 \left(2\langle \hat{E}_0 \rangle^2 \langle \hat{\mathrm{s}}_0 - \rho \rangle - 2\langle \hat{E}_0 \rangle \langle \hat{E}_0 (\hat{\mathrm{s}}_0 - \rho) \rangle - \langle \hat{E}_0^2 \rangle \langle \hat{\mathrm{s}}_0 - \rho \rangle + \langle \hat{E}_0^2 \, \hat{\mathrm{s}}_0 - \rho \rangle \right) \\ & + 6\Lambda m_b^5 \Big(-2\langle \hat{\mathrm{s}}_0 - \rho \rangle \langle \hat{E}_0 (\hat{\mathrm{s}}_0 - \rho) \rangle \\ & \quad + \langle \hat{E}_0 \rangle \left(2\langle \hat{\mathrm{s}}_0 - \rho \rangle^2 - \langle (\hat{\mathrm{s}}_0 - \rho)^2 \rangle \right) + \langle \hat{E}_0 (\hat{\mathrm{s}}_0 - \rho)^2 \rangle \Big) \\ & + 8\Lambda^3 m_b^3 \left(2\langle \hat{E}_0 \rangle^3 - 3\langle \hat{E}_0^2 \rangle \langle \hat{E}_0 \rangle + \langle \hat{E}_0^3 \rangle \right) \\ & + m_b^6 \left(2\langle \hat{\mathrm{s}}_0 - \rho \rangle^3 - 3\langle (\hat{\mathrm{s}}_0 - \rho)^2 \rangle \langle \hat{\mathrm{s}}_0 - \rho \rangle + \langle (\hat{\mathrm{s}}_0 - \rho)^3 \rangle \right) . \end{aligned} \tag{6.28}$$

but here with a cut on the electron energy (ξ) for both, the moment *and* the normalization:

$$\langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle = \frac{\langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle (\xi)}{\Gamma^{(0,0)}(\xi)}$$

for every expectation value, where $\langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle (\xi)$ is only normalized to the common prefactor $\frac{G_{\mathrm{F}}^2 |V_{cb}|^2 m_b^5}{192\pi^3}$.

All partonic moments, the *building blocks*, have their perturbative and non-perturbative expansion and a term for the change of the mass scheme from pole mass to kinetic mass:

$$\begin{aligned} \langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle (\xi) = {} & \langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle^{(0,0)} (\xi) + \frac{\mathrm{d}\langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle^{(0,0)} (\xi)}{\mathrm{d}m_{\mathrm{b}}^{\mathrm{kin}}} \delta m_{\mathrm{b}}^{\mathrm{kin}} + \frac{\mathrm{d}\langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle^{(0,0)} (\xi)}{\mathrm{d}m_{\mathrm{c}}^{\mathrm{kin}}} \delta m_{\mathrm{c}}^{\mathrm{kin}} \\ & + \frac{\alpha_s}{\pi} \langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle^{(0,1)} (\xi) + \frac{\hat{\mu}_\pi^2}{m_{\mathrm{b}}^2} \langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle^{(2,0)}_{\hat{\mu}_\pi^2} (\xi) + \frac{\hat{\mu}_{\mathrm{G}}^2}{m_{\mathrm{b}}^2} \langle \hat{E}_0^i (\hat{\mathrm{s}}_0 - \rho)^j \rangle^{(2,0)}_{\hat{\mu}_{\mathrm{G}}^2} (\xi) + \dots \end{aligned} \tag{6.29}$$

with

$$\delta m_{\mathrm{b}}^{\mathrm{kin}} (\mu_f) = \frac{\alpha_s}{\pi} \left(\frac{16}{9} \mu_f + \frac{2\mu_f^2}{3m_{\mathrm{b}}} \right), \tag{6.30}$$

$$\delta m_{\mathrm{c}}^{\mathrm{kin}} (\mu_f) = \frac{\alpha_s}{\pi} \left(\frac{16}{9} \mu_f + \frac{2\mu_f^2}{3m_{\mathrm{c}}} \right). \tag{6.31}$$

as for the lepton energy moments.

6.2.4 Experimental Input

The combined fit is performed with a selection of the following 25 moment measurements by *BABAR* which are characterized by correlations below 95 % to ensure the invertibility of the covariance matrix:

- Lepton energy moments measured by *BABAR* [36]. We use the partial branching fraction $\mathcal{B}_{p^*_{\ell,\mathrm{min}}}$ at the minimal lepton momentum $p^*_\ell \geq 0.6,1.0,1.5\,\mathrm{GeV}/c$, the moments $\langle E_\ell \rangle$ for $p^*_\ell \geq 0.6,0.8,1.0,1.2,1.5\,\mathrm{GeV}/c$, the central moments $\langle (E_\ell - \langle E_\ell \rangle)^2 \rangle$ for $p^*_\ell \geq 0.6,1.0,1.5\,\mathrm{GeV}/c$ and $\langle (E_\ell - \langle E_\ell \rangle)^3 \rangle$ for $p^*_\ell \geq 0.8,1.2\,\mathrm{GeV}/c$.
- Hadronic mass moments measured by *BABAR* [34]. We use the moment $\langle m_X^2 \rangle$ for $p^*_\ell \geq 0.9,1.1,1.3,1.5\,\mathrm{GeV}/c$ and the central moments $\langle (m_X^2 - \langle m_X^2 \rangle)^2 \rangle$ and $\langle (m_X^2 - \langle m_X^2 \rangle)^3 \rangle$ both for $p^*_\ell \geq 0.8,1.0,1.2, 1.4\,\mathrm{GeV}/c$.

Furthermore we use the average B meson lifetime $\tau_B = f_0\tau_0 + (1 - f_0)\tau_\pm = (1.585 \pm 0.007)\,\mathrm{ps}$ with the lifetimes of neutral and charged B mesons τ_0 and $\tau_\pm$ and the relative production rate, $f_0 = 0.491 \pm 0.007$, as quoted in [37].

6.2.5 Theoretical Uncertainties

Theoretical uncertainties for the prediction of the moments $\overrightarrow{M}_{\mathrm{theo}}$ are estimated by variation of the parameters. The standard model parameters, that are all exept c'_{R}, are treated as in [34]. The uncertainty in the non-perturbative part are estimated by varying the corresponding parameters μ_π^2 and μ_G^2 by 20 % and ρ_D^3 and ρ_{LS}^3 by 30 % around their expected value. For the uncertainties of the perturbative corrections $\alpha_S = 0.22$ is varied up and down by 0.1 for the hadronic mass moments and 0.04 for the lepton energy moments and the uncertainties of the perturbative correction of the quark masses m_b and m_c are estimated by varying them $20\,\mathrm{MeV}/c^2$ up and down. An additional error of 1.4 % is added to $|V_{cb}|$ from the fit for the uncertainty in the expansion of the semileptonic rate Γ_{SL}, which is not included in the fit, but quoted separately as theoretical uncertainty on $|V_{cb}|$.

Additionally the influence of the right-handed contributions on the theoretical uncertainties in the predictions of the moments has to be included. Varying c'_{R} in a similar fashion as the other parameters, around the a priory estimate of zero showed only very little influence on the fit results. Due to the fact that the right-handed contributions are included up to $1/m_b^2$ in the non-perturbative and $\mathcal{O}(\alpha_s)$ in the perturbative corrections for all moments, the uncertainties in the prediction of the moments are not sizable and thus the variation of c'_{R} has to be rather small. For the final results the variation of c'_{R} has not been included, because of no influence on the significant digits.

6.3 Results

Table 6.1 shows the fit results and table 6.2 the corresponding standard-model fit results, which were obtained by performing the fit with c'_{R} fixed to zero. Figs. 6.1 and 6.2 show a comparison of the fit results with the measured moments for the lepton-moments and the hadronic-mass moments, respectively. The uncertainties Δ_{exp} and Δ_{theor} are the expected experimental and theory

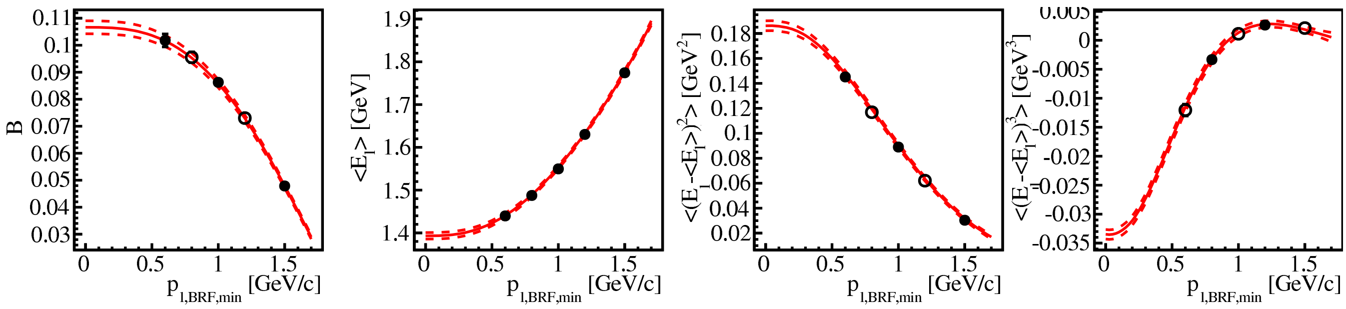


Figure 6.1: *The measured lepton-moments (•/○) compared with the result of the simultaneous fit (solid red line) as function of the minimal lepton momentum $p^*_{\ell,\min}$. The measurements included in the fit are marked by solid data points (•). The dashed lines indicate the theoretical fit uncertainty obtained by the variation of the fit parameters in order to convert their theoretical uncertainty into an error of the moments.*

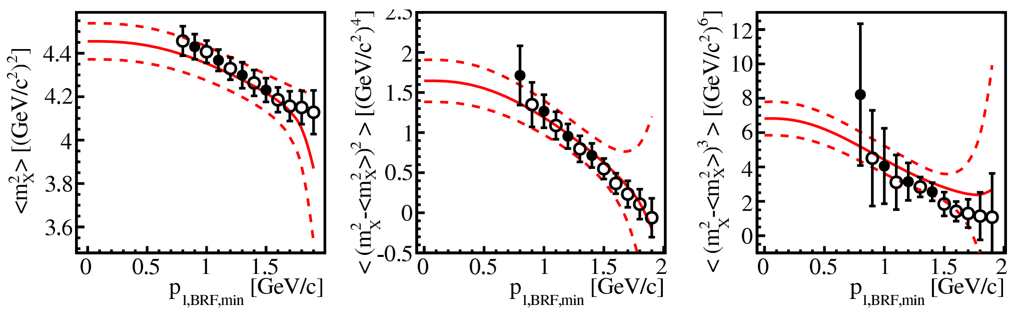


Figure 6.2: *The measured hadronic-mass moments (•/○) compared with the result of the simultaneous fit (solid red line) as function of the minimal lepton momentum $p^*_{\ell,\min}$. The measurements included in the fit are marked by solid data points (•). The dashed lines indicate the theoretical fit uncertainty obtained by the variation of the fit parameters in order to convert their theoretical uncertainty into an error of the moments.*

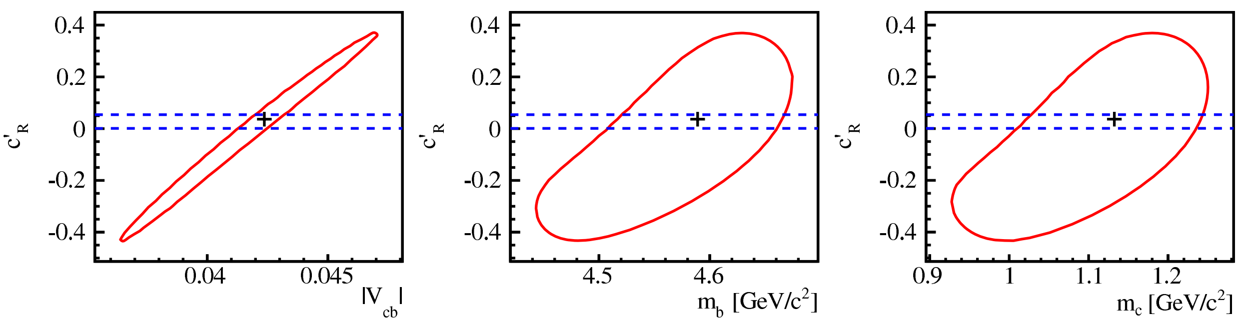


Figure 6.3: *The $\Delta\chi^2 = 1$ contours in the (c'_R,$|V_{cb}|$), (c'_R,m_b) and (c'_R,m_c) plane for the results obtained in the fit. The blue dashed lines show the uncertainty of c'_R from exclusive decays ($c'_R = 0.014^{+0.040}_{-0.013}$) as computed in section 6.4.*

Table 6.1: *Results of the full fit for c'_R and the canonical set of parameters $|V_{cb}|$, m_b, m_c, $\mathcal{B}$, μ_π^2, μ_G^2, ρ_D^3 and ρ_{LS}^3, separated by experimental and theoretical uncertainties. For $|V_{cb}|$ we take into account an additional error of 1.4 % for the uncertainty in the expansion of the semileptonic rate Γ_{SL}. Correlation coefficients for the parameters are listed below. The uncertainties Δ_{exp} and Δ_{theor} are the expected experimental and theory errors determined by Toy-MC studies (see [38]) while Δ_{tot} is the total uncertainty provided by the fit.*

	c'_R	$\lvert V_{cb}\rvert$ $\times 10^{-3}$	m_b [GeV/c^2]	m_c [GeV/c^2]	$\mathcal{B}$ [%]	μ_π^2 [GeV^2]	μ_G^2 [GeV^2]	ρ_D^3 [GeV^3]	ρ_{LS}^3 [GeV^3]
Results	0.0366	42.37	4.589	1.132	10.665	0.466	0.299	0.201	-0.126
Δ_exp	0.1313	1.94	0.045	0.071	0.214	0.031	0.045	0.013	0.076
Δ_theo	0.4006	5.26	0.098	0.129	0.110	0.092	0.042	0.057	0.060
$\Delta_{\Gamma_{SL}}$		0.59							
Δ_tot	$^{+0.3329}_{-0.4693}$	$^{+4.69}_{-5.99}$	$^{+0.085}_{-0.146}$	$^{+0.118}_{-0.205}$	$^{+0.235}_{-0.253}$	$^{+0.112}_{-0.086}$	$^{+0.061}_{-0.063}$	$^{+0.083}_{-0.044}$	$^{+0.096}_{-0.097}$
c'_R	1.00	0.99	0.69	0.63	0.60	-0.74	0.26	-0.83	0.16
$\lvert V_{cb}\rvert$		1.00	0.65	0.60	0.66	-0.71	0.21	-0.81	0.16
m_b			1.00	0.98	0.55	-0.66	0.23	-0.64	0.11
m_c				1.00	0.56	-0.65	0.09	-0.60	0.15
$\mathcal{B}$					1.00	-0.37	0.09	-0.46	0.10
μ_π^2						1.00	-0.15	0.88	-0.06
μ_G^2							1.00	-0.23	0.01
ρ_D^3								1.00	-0.24
ρ_{LS}^3									1.00

Table 6.2: *Results of the standard model fit with c'_R fixed to zero.*

	$\lvert V_{cb}\rvert$ $\times 10^{-3}$	m_b [GeV/c^2]	m_c [GeV/c^2]	$\mathcal{B}$ [%]	μ_π^2 [GeV^2]	μ_G^2 [GeV^2]	ρ_D^3 [GeV^3]	ρ_{LS}^3 [GeV^3]
Results	41.88	4.582	1.124	10.651	0.473	0.298	0.205	-0.128
Δ_exp	0.43	0.057	0.084	0.175	0.022	0.040	0.014	0.080
Δ_theo	0.40	0.051	0.075	0.058	0.056	0.044	0.026	0.051
$\Delta_{\Gamma_{SL}}$	0.59							
Δ_tot	$^{+0.83}_{-0.84}$	$^{+0.079}_{-0.075}$	$^{+0.112}_{-0.114}$	$^{+0.185}_{-0.185}$	$^{+0.060}_{-0.061}$	$^{+0.059}_{-0.059}$	$^{+0.030}_{-0.030}$	$^{+0.095}_{-0.095}$
$\lvert V_{cb}\rvert$	1.00	-0.40	-0.26	0.65	0.33	-0.39	0.31	0.07
m_b		1.00	0.98	0.24	-0.32	0.09	-0.17	0.01
m_c			1.00	0.28	-0.35	-0.08	-0.19	0.06
$\mathcal{B}$				1.00	0.14	-0.08	0.09	0.01
μ_π^2					1.00	0.06	0.71	0.09
μ_G^2						1.00	-0.03	-0.03
ρ_D^3							1.00	-0.20
ρ_{LS}^3								1.00

errors determined by Toy-MC studies (see [38]) while $\Delta_{\rm tot}$ is the total uncertainty provided by the fit.

The estimate for $c'_{\rm R} = 0.04^{+0.33}_{-0.47}$ is consistent with the standard-model prediction of zero, but the uncertainty reveals an unexpected low sensitivity of the semileptonic fit on possible right-handed contributions. We state the upper relative admixture limit of 0.9 at 95 % confidence level.

The extracted value of $|V_{cb}| = (42^{+5}_{-6}) \cdot 10^{-3}$ is consistent with the value from the standard-model fit, but its uncertainty is quite different. In our fit, this is due to the influence of a sizable $c'_{\rm R}$ uncertainty on the determination of $|V_{cb}|$ in (6.10), which becomes evident in the contour plot of the ($c'_{\rm R}$,$|V_{cb}|$) plane, showing a shallow and steep covariance ellipse.

To compare the quality of the fits the P-value (prob($\chi^2, n_{\rm dof}$)) suits best, because the fits differ by their number of degrees of freedom and thus the χ^2 value alone is not sufficient. For the fit with $c'_{\rm R}$ we find $\chi^2 = 7.390$ with 17 degrees of freedom and thus prob$(7.390, 17) = 0.978$ and for the standard-model fit prob$(7.397, 18) = 0.986$, which shows neither improvement nor worsening.

The uncertainty of the result for $c'_{\rm R}$ is dominated by the theory error $\Delta_{\rm theo}$=0.40 in table 6.1 compared to $\Delta_{\rm exp}$=0.14. As a consequence including additional experimental data, e. g. from Belle, will not improve the limit for a possible right-handed contribution at this point. On the other hand the theory prediction for the right-handed part is performed up to the order $1/m_b^2$ in the non-perturbative correction and up to order α_s in the perturbative correction. Compared to the standard-model prediction only the $1/m_b^3$ and BLM corrections are missing and the term quadratic in the right-handed contribution. Due to the smallness of these missing corrections they would not decrease our limit significantly. The shape of the considered spectra and hence their moments are too similar for a left and right-handed $b \to c$ current, ending up in a low sensitivity of $c'_{\rm R}$ and a weak constrain therein.

6.4 Right Handed Admixture from Exclusive Decays

It is interesting to note that the value of $|V_{cb}|$ extracted in this way $|V_{cb}| = (42^{+5}_{-6}) \cdot 10^{-3}$ is in agreement with $|V_{cb}|$ from exclusive decays, from which $|V_{cb}| = (38.6 \pm 1.3) \cdot 10^{-3}$ (from $\overline{B} \to D^* \ell^- \nu$) [39] is obtained, while the value from the standard-model fit $|V_{cb}| = (41.88 \pm 0.59) \cdot 10^{-3}$ is not. This is due to the low sensitivity of $c'_{\rm R}$ and thus the large uncertainty of the extracted value. In addition, also the exclusive decays allow us to constrain a possible right-handed admixture [40, 41].

The most straightforward way of obtaining this information is to study the exclusive differential rates at the point of maximal momentum transfer to the leptons, corresponding to equal four-velocities of the initial and final hadron. We consider the decays $\overline{B} \to {\rm d}\ell^- \nu$ and $\overline{B} \to D^* \ell^- \nu$. The corresponding rates in the standard model are usually parametrized in terms of two form factors; the relevant expressions close to the point of maximal momentum transfer read

$$\begin{aligned} \frac{{\rm d}\Gamma^{B \to D}}{{\rm d}w} &= \Gamma_0 16 r^3 (r+1)^2 (w^2-1)^{3/2} (|V_{cb}| \mathcal{G}(w))^2, \\ \frac{{\rm d}\Gamma^{B \to D^*}}{{\rm d}w} &= \Gamma_0 192 r_*^3 (r_* - 1)^2 (w^2-1)^{1/2} (|V_{cb}| \mathcal{F}(w))^2 \end{aligned} \qquad (6.32)$$

where w=$v \cdot v'$ is the scalar product of the hadronic velocities, r=m_D/m_B, r_*=m_{D^*}/m_B, and Γ_0=$G_F^2 m_B^5/(192\pi^3)$.

The information extracted by the experiments in the context of the $|V_{cb}|^2$ determination is

$$\lim_{w \to 1} \frac{\mathrm{d}\Gamma^{B \to D}}{\mathrm{d}w} \frac{1}{\Gamma_0 16 r^3 (r+1)^2 (w^2-1)^{3/2}},$$
$$\lim_{w \to 1} \frac{\mathrm{d}\Gamma^{B \to D^*}}{\mathrm{d}w} \frac{1}{\Gamma_0 192 r_*^3 (r_*-1)^2 (w^2-1)^{1/2}} \tag{6.33}$$

which in the standard model is the product of the form factors at $w = 1$ and $|V_{cb}|$. Combining this with a theoretical prediction of the form factors at $w = 1$ one extracts $|V_{cb}|$.

At the non-recoil point $w = 1$ the $B \to D$ transition is completely dominated by the vector current, while the $B \to D^*$ decay is proportional to the axial vector current. Thus, including a right-handed admixture, the information extracted from (6.33) is $|c_\mathrm{L} + c_\mathrm{R}||V_{cb}|\mathcal{G}(1)$ for the case of the $B \to D$ transition and $|c_\mathrm{R} - c_\mathrm{L}||V_{cb}|\mathcal{F}(1)$ for $B \to D^*$. The current experimental data yield [42]:

$$|c_\mathrm{L} + c_\mathrm{R}||V_{cb}|\mathcal{G}(1) = (42.4 \pm 1.56) \times 10^{-3} \tag{6.34}$$
$$|c_\mathrm{R} - c_\mathrm{L}||V_{cb}|\mathcal{F}(1) = (35.41 \pm 0.52) \times 10^{-3} \tag{6.35}$$

Using the lattice data (which are also used to extract $|V_{cb}|$) [43–45]

$$\mathcal{G}(1) = 1.074 \pm 0.024 \tag{6.36}$$
$$\mathcal{F}(1) = 0.921 \pm 0.025 \tag{6.37}$$

we can extract the ratio $c'_\mathrm{R} = c_\mathrm{R}/c_\mathrm{L}$ to be

$$c'_\mathrm{R} = 0.01 \pm 0.03 \tag{6.38}$$

with the assumption of no sizable correlations between the experimental measurements of the right-hand sides of Eqns. (6.34) and (6.35) as well as between the form factor values given in (6.36) and (6.37). The value for $|V_{cb}|$ extracted from Eqns. (6.34) and (6.35) is found to be $|V_{cb}|_\mathrm{excl} = (39.1^{+1.4}_{-1.1}) \cdot 10^{-3}$ which has to be compared to $|V_{cb}|_\mathrm{excl} = (38.8 \pm 1.0) \cdot 10^{-3}$ when setting $c'_\mathrm{R} = 0$ in Eqns. (6.34) and (6.35).

The result of c'_R is compatible with zero and, in fact, more restrictive than the determination from inclusive decays. Obviously the exclusive decay gives access to data separated by the handedness of the $b \to c$ current in contrast to the inclusive decay, leading to a better limit on possible right-handed contributions.

In turn, we can use the result for c'_R to determine $|V_{cb}|_\mathrm{incl}$ and compare with $|V_{cb}|_\mathrm{excl}$. This can be done by imposing a Gaussian constraint of c'_R=0.01±0.03 in the fit with a possible right handed current. The result $|V_{cb}|_\mathrm{incl}$=(42.0±0.9)$\cdot 10^{-3}$ compared to $|V_{cb}|_\mathrm{excl}$=$(39.1^{+1.4}_{-1.1})\cdot 10^{-3}$ exhibits a tension by 2.9 of the central values. Determing $|V_{cb}|$ by (6.36) and (6.37) with c'_R set to zero gives $|V_{cb}|_\mathrm{excl}(c'_\mathrm{R}=0)$=(38.8±1.0)$\cdot 10^{-3}$ and allows us to examine the differences in the tensions between inclusive and exclusive decays with and without a right-handed current, by comparing this value to the standard model fit value $|V_{cb}|_\mathrm{incl}(c'_\mathrm{R}=0)$=(41.9±0.8)$\cdot 10^{-3}$ (see table 6.2) yielding a tension of about 3.1 of the central values. As a consequence, the difference in the central values between $|V_{cb}|$ exclusive and inclusive is slightly reduced, and more importantly, the uncertainty on $|V_{cb}|$ exclusive is considerably larger when allowing for a right-handed admixture resulting in a smaller significance of the observed deviation. In our analysis, which is using only the inclusive BABAR data, the difference between exclusive and inclusive is reduced from a $2.4\,\sigma$ to a $1.7\,\sigma$ deviation.

7 The α_s/m_b^2 Corrections to the Inclusive Decay $\overline{B} \longrightarrow X_c \ell \overline{\nu}_\ell$

7.1 Correction to μ_π^2 with Reparametrization Invariance

A theory has a reparametrization invariance when its Hamiltonian is invariant by a shift of the momenta involved. This is the case for the studied decay, which allows us to shift the quark momenta by the small residual momentum k of the b quark in the B meson and then Taylor expand it accordingly in order to express it in terms of the original system.

The studied decay at parton level is calculated in the rest frame of the decaying b quark: $p_\mathrm{b} = m_\mathrm{b} \cdot v$. We shift the system by the small residual momentum k, meaning the shift $p'_\mathrm{b} = p_\mathrm{b} + k$ and $p'_\mathrm{c} = p_\mathrm{c} + k$. Because we are only interested in various moment of lepton energy spectra and the hadronic invariant mass, the calculation of the parametrized system turns out to be fairly simple. As an first example consider the full decay rate at parton and tree level:

$$\Gamma = \frac{1}{2m_b}\left(\prod_{f=c,e,\nu}\int\frac{\mathrm{d}^3p_f}{(2\pi)^3}\frac{1}{2E_f}\right)\left|\mathcal{M}(m_b \to \{p_f\})\right|^2(2\pi)^4\delta^4\left(p_b - \sum p_f\right)$$

$$\Gamma^{(0,0)} = \frac{G_\mathrm{F}^2|V_{cb}|^2 m_b^5}{192\pi^3}\, f(\rho) \qquad \text{with} \qquad f(\rho) = \left(1 - 8\rho - 12\rho^2\ln\rho + 8\rho^3 - \rho^4\right)$$

Due to the phase space integration there is no momentum left to reparametrize except one: The origin of the normalizing prefactor $\frac{1}{2m_b}$ is $\frac{1}{2E_b}$, with the energy of the b quark in its rest frame $p_\mathrm{b} = m_\mathrm{b} v = (m_\mathrm{b}, 0, 0, 0)$.

Transformation to the system with residual momentum

$$m_b v \longrightarrow m_b v + k$$

with $v^\alpha = (1, 0, 0, 0)$, yields a time dilation factor:

$$\frac{1}{2m_b} \longrightarrow \frac{1}{2(m_b v_0 + k_0)} = \frac{1}{2m_b}\frac{1}{1 + \frac{v \cdot k}{m_b}}$$

We expand the time dilation factor up to second order in $v \cdot k$

$$\frac{1}{1 + \frac{v \cdot k}{m_b}} \;\to\; 1 - \frac{v \cdot k}{m_b} + \frac{(v \cdot k)^2}{m_b^2} \tag{7.1}$$

and perform an averaging of the residual momentum k:

$$\langle k^\alpha \rangle = \frac{\mu_\pi^2}{2m_b} v^\alpha \qquad \langle k^\alpha k^\beta \rangle = -\frac{\mu_\pi^2}{3}(g^{\alpha\beta} - v^\alpha v^\beta). \tag{7.2}$$

With $v^2 = 1$ the quadratic term in (7.1) vanishes:

$$(v \cdot k)^2 = v_\alpha v_\beta k^\alpha k^\beta = -\frac{\mu_\pi^2}{3} v_\alpha v_\beta (g^{\alpha\beta} - v^\alpha v^\beta) = 0,$$

the time dilation factor becomes

$$\frac{1}{1 + \frac{v \cdot k}{m_b}} \;\to\; 1 - \frac{\mu_\pi^2}{2m_b^2}$$

and hence the decay rate:

$$\frac{G_\mathrm{F}^2 |V_{cb}|^2 m_b^5}{192\pi^3} f(\rho) \;\to\; \frac{G_\mathrm{F}^2 |V_{cb}|^2 m_b^5}{192\pi^3} \left(1 - \frac{\mu_\pi^2}{2m_b^2}\right) f(\rho). \tag{7.3}$$

7.1.1 Moments of the Lepton Energy Spectrum

The application of the reparametrization invariance to the moments is similar, but we have to account for the weight of the integral, which has to be reparametrized as well, because it is responsible for the dimension of the integral and thus its transformation under the boost. Moreover, the moments are normalized to the decay rate and we have to include its transformation we just calculated.

The moments of the lepton energy spectrum are defined as

$$L_n = \frac{1}{\Gamma_0} \int_{E_\mathrm{cut}} \mathrm{d}E_\ell \, E_\ell^n \, \frac{\mathrm{d}\Gamma}{\mathrm{d}E_\ell}$$

For the reparametrization the decay rate Γ_0 as a normalizing factor is not included, because it is just used as a relative numerical basis to express the moments. We have to consider $\mathrm{d}\Gamma$ which transforms like (7.3) and a reparametrization of E_ℓ^n. The reparametrizations of both $\mathrm{d}E_\mathrm{l}$s cancel.

For $n = 0$ we gain the transformed decay rate which is just numerically normalized to the untransformed decay rate. In the case of $n = 1$ the lepton energy E_l to the power of one is reparametrized as

$$E_\mathrm{l} = v \cdot p_\mathrm{l} = \frac{p_\mathrm{b}}{m_\mathrm{b}} \cdot p_\mathrm{l} \;\to\; \left(v + \frac{k}{m_\mathrm{b}}\right) \cdot p_\mathrm{l} = \left(E_\ell + \frac{k \cdot p_\ell}{m_b}\right)$$

and averaged according to (7.2):

$$\left\langle E_\ell + \frac{k \cdot p_\ell}{m_b} \right\rangle = E_\ell \left(1 + \frac{\mu_\pi^2}{2m_b^2}\right).$$

This cancels directly with the time dilation factor $1/(1 + \frac{v \cdot k}{m_b})$ from $\mathrm{d}\Gamma$ without expanding that to some order. Hence the $1/m_b^2$ correction to L_1 vanishes.

For the calculation with $n = 2$ we have to consider the transformation of E_{l}^2 and $\mathrm{d}\Gamma$:

$$
\begin{aligned}
E_{\mathrm{l}} &\rightarrow \left(v \cdot p_{\mathrm{l}} + \frac{k \cdot p_{\mathrm{l}}}{m_{\mathrm{b}}}\right)^2 = E_{\mathrm{l}}^2 + 2E_{\mathrm{l}}\frac{k \cdot p_{\mathrm{l}}}{m_{\mathrm{b}}} + \frac{(k \cdot p_{\mathrm{l}})^2}{m_{\mathrm{b}}^2} \\
\mathrm{d}\Gamma &\rightarrow \frac{\mathrm{d}\Gamma}{1 + \frac{v \cdot k}{m_{\mathrm{b}}}} = 1 - \frac{v \cdot k}{m_{\mathrm{b}}} + \frac{(v \cdot k)^2}{m_{\mathrm{b}}^2},
\end{aligned}
$$

and especially the product, which transforms as:

$$
E_{\mathrm{l}}^2 \mathrm{d}\Gamma \rightarrow \left(E_{\mathrm{l}}^2 + 2E_{\mathrm{l}}\frac{k \cdot p_{\mathrm{l}}}{m_{\mathrm{b}}} + \frac{(k \cdot p_{\mathrm{l}})^2}{m_{\mathrm{b}}^2}\right)\left(1 - \frac{v \cdot k}{m_{\mathrm{b}}} + \frac{(v \cdot k)^2}{m_{\mathrm{b}}^2}\right)\mathrm{d}\Gamma
$$

and by expanding the product up to the order k^2:

$$
= \left(E_{\mathrm{l}}^2 + 2E_{\mathrm{l}}\frac{k \cdot p_{\mathrm{l}}}{m_{\mathrm{b}}} - E_{\mathrm{l}}^2\frac{v \cdot k}{m_{\mathrm{b}}} - 2E_{\mathrm{l}}\frac{(v \cdot k)(k \cdot p_{\mathrm{l}})}{m_{\mathrm{b}}^2} + \frac{(k \cdot p_{\mathrm{l}})^2}{m_{\mathrm{b}}^2} + \frac{(v \cdot k)^2}{m_{\mathrm{b}}^2}E_{\mathrm{l}}^2\right)\mathrm{d}\Gamma
$$

and averaging it according to (7.2):

$$
\begin{aligned}
&= \left(E_{\mathrm{l}}^2 + 2E_{\mathrm{l}}\frac{\mu_\pi^2}{2m_{\mathrm{b}}} + E_{\mathrm{l}}^2\frac{\mu_\pi^2}{3m_{\mathrm{b}}^2} - E_{\mathrm{l}}^2\frac{\mu_\pi^2}{2m_{\mathrm{b}}}\right)\mathrm{d}\Gamma \\
&= \left(1 + \frac{5}{3}\frac{\mu_\pi^2}{2m_{\mathrm{b}}}\right)E_{\mathrm{l}}^2\mathrm{d}\Gamma
\end{aligned}
$$

yields a factor of 5/3 for the $\frac{\mu_\pi^2}{2m_b^2}$ coefficient of the L_2 moment of the lepton energy spectrum. The calculation of L_3 is analog to the one of L_2 and gives a corresponding factor of 4. We collect the results in table 7.1. The table lists the weights of the integral L_n' after reparametrization.

n	$\mathbf{L_n'}$ **weight**
0	$E_{\mathrm{l}}\left(1 - \frac{\mu_\pi^2}{2m_b^2}\right)$
1	E_{l}^2
2	$E_{\mathrm{l}}^3\left(1 + \frac{5}{3}\frac{\mu_\pi^2}{2m_b^2}\right)$
3	$E_{\mathrm{l}}^4\left(1 + 4\frac{\mu_\pi^2}{2m_b^2}\right)$

Figure 7.1: *Reparametrized weights for the moments of the lepton energy spectrum*

The corresponding calculation for the α_s/m_b^2 corrections of the moments is trivial, because the argumentation with the time dilation factor and the reparametrization of the weights of the integrals is independent from the order of α_s in the calculation. Hence the weights in table 7.1 are the same for the reparametrized $\mathcal{O}(\alpha_s)$ weights of the moment L_n.

7.1.2 Moments of the hadronic invariant mass

Using the reparametrization invariance of the moments of the hadronic invariant mass to compute the kinetic corrections follows the same strategy as for the leptonic moments:

- Shift the quark momenta $p'_\mathrm{b} = p_\mathrm{b} + k$ and $p'_\mathrm{c} = p_\mathrm{c} + k$ and thus the hadronic energy

  $$E_\mathrm{x} = v \cdot p_\mathrm{x} \to (v + k/m_\mathrm{b})(p_\mathrm{x} + k) = v \cdot p_\mathrm{x} + v \cdot k + k \cdot p_\mathrm{x}/m_\mathrm{b} + k^2/mb$$

  and the hadronic invariant mass

  $$p_\mathrm{x} \to (p_\mathrm{x} + k)^2 = p_\mathrm{x}^2 + 2p_\mathrm{x} \cdot k + k^2.$$

- Include the time dilation factor of $\mathrm{d}\Gamma$ and reparametrize $(p_\mathrm{x})^i E_\mathrm{x}^j \, \mathrm{d}\Gamma$
- Expand up to the second order in k.
- Average the expanded expression according to

  $$\langle k^\alpha \rangle = \frac{\mu_\pi^2}{2m_b} v^\alpha \qquad \langle k^\alpha k^\beta \rangle = -\frac{\mu_\pi^2}{3}(g^{\alpha\beta} - v^\alpha v^\beta).$$

The results are presented in table 7.2 in form of the reparametrized weight of the moments of the hadronic invariant mass.

i	j	$\mathbf{M_{i,j}}$ weights
0	1	$E_\mathrm{x} + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}(-m_\mathrm{b})$
0	2	$E_\mathrm{x}^2 + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(\frac{5}{3}E_\mathrm{x}^2 - 2m_\mathrm{b}E_\mathrm{x} - \frac{2}{3}p_\mathrm{x}^2\right)$
0	3	$E_\mathrm{x}^3 + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(4E_\mathrm{x}^3 - 3m_\mathrm{b}E_\mathrm{x}^2 - 2p_\mathrm{x}^2 E_\mathrm{x}\right)$
1	0	$p_\mathrm{x}^2 + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(-2m_\mathrm{b}^2 + 2E_\mathrm{x}m_\mathrm{b} - p_\mathrm{x}^2\right)$
2	0	$\left(p_\mathrm{x}^2\right)^2 + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(\frac{8}{3}E_\mathrm{x}^2 m_\mathrm{b}^2 - \frac{20}{3}p_\mathrm{x}^2 m_\mathrm{b}^2 + 4E_\mathrm{x}p_\mathrm{x}^2 m_\mathrm{b} - \left(p_\mathrm{x}^2\right)^2\right)$
3	0	$\left(p_\mathrm{x}^2\right)^3 + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(-\left(p_\mathrm{x}^2\right)^3 - 14m_\mathrm{b}^2\left(p_\mathrm{x}^2\right)^2 + 6E_\mathrm{x}m_\mathrm{b}\left(p_\mathrm{x}^2\right)^2 + 8E_\mathrm{x}^2 m_\mathrm{b}^2 p_\mathrm{x}^2\right)$
1	1	$p_\mathrm{x}^2 E_\mathrm{x} + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(\frac{10}{3}m_\mathrm{b}E_\mathrm{x}^2 - 2m_\mathrm{b}^2 E_\mathrm{x} - \frac{7}{3}m_\mathrm{b}p_\mathrm{x}^2\right)$
2	1	$\left(p_\mathrm{x}^2\right)^2 E_\mathrm{x} + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(\frac{8}{3}m_\mathrm{b}^2 E_\mathrm{x}^3 + \frac{20}{3}m_\mathrm{b}p_\mathrm{x}^2 E_\mathrm{x}^2 - \frac{20}{3}m_\mathrm{b}^2 p_\mathrm{x}^2 E_\mathrm{x} - \frac{11}{3}m_\mathrm{b}\left(p_\mathrm{x}^2\right)^2\right)$
1	2	$p_\mathrm{x}^2 E_\mathrm{x}^2 + \frac{\mu_\pi^2}{2m_\mathrm{b}^2}\left(\frac{14}{3}m_\mathrm{b}E_\mathrm{x}^3 - 2m_\mathrm{b}^2 E_\mathrm{x}^2 + \frac{5}{3}p_\mathrm{x}^2 E_\mathrm{x}^2 - \frac{14}{3}m_\mathrm{b}p_\mathrm{x}^2 E_\mathrm{x} - \frac{2}{3}\left(p_\mathrm{x}^2\right)^2\right)$

Figure 7.2: *Reparametrized weights for the moments of the hadronic invariant mass*

As discussed earlier the HQE parameter μ_π^2 is the kinetic energy of the b quark within the B meson. As we have seen this kinetic correction does not depend on the certain decay structure, but is simply a boost of the b quark rest frame to the more realistic frame where the B meson is at rest and the b quark has a small residual momentum.

Unfortunately the coefficient of the second important HQE parameter $\hat{\mu}_{\mathrm{G}}^2$ at $\mathcal{O}(1/m_{\mathrm{b}}^2)$ cannot be calculated by means of reparametrization invariance. As can be seen from its definition

$$
\begin{aligned}
-\mu_\pi^2 &= \langle \bar{B}|\bar{b}_v(i\mathrm{D})^2 b_v|\bar{B}\rangle \\
\mu_G^2 &= \langle \bar{B}|\bar{b}_v \tfrac{g}{2}\sigma_{\mu\nu}G^{\mu\nu} b_v|\bar{B}\rangle
\end{aligned}
$$

the parameter $\hat{\mu}_{\mathrm{G}}^2$ depends on the spin structure and involves one-gluon matrix elements. The averaging procedure

$$
\langle k^\alpha\rangle = \frac{\mu_\pi^2}{2m_b}v^\alpha \qquad \langle k^\alpha k^\beta\rangle = -\frac{\mu_\pi^2}{3}(g^{\alpha\beta} - v^\alpha v^\beta)
$$

does *not* preserve the order of the covariant derivatives an hence does not reproduce the contributions of the one-gluon matrix elements. The correction to the matrix elements μ_π^2 could be calculated with reparametrization invariance, because it is symmetric and does not contain any gluon-matrix element.

7.2 Full Calculation for μ_π^2

For a full calculation of the $\mathcal{O}(\alpha_s)$ corrections to μ_π^2 we can combine the approaches of the HQE and radiative corrections. First we consider the same Feynman diagrams as for the radiative corrections and expand all occurring propagators similar to 4.5. In the real and virtual corrections there are three different types of propagators:

$$
\frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}}, \qquad \frac{1}{\not{p}_{\mathrm{b}} - \not{p}_{\mathrm{g}} - m_{\mathrm{b}}} \quad \text{and} \quad \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}}. \tag{7.4}
$$

where the c-quark momentum is $p_{\mathrm{c}} = p_{\mathrm{b}} - q = m_{\mathrm{b}}v - q$. Introducing the residual momentum k amounts for the replacement $p_{\mathrm{b}} \to p_{\mathrm{b}} + k = m_{\mathrm{b}}v + \mathrm{i}\not{D}$ and $p_{\mathrm{c}} \to p_{\mathrm{c}} + k = m_{\mathrm{b}}v + \mathrm{i}\not{D} - q$. Expanding the propagators to second order in iD in the same fashion as 4.5 yields:

$$
\begin{aligned}
\frac{i}{\not{p}_{\mathrm{c}} + \mathrm{i}\not{D} - m_{\mathrm{c}}} =& \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} - \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} + \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} - \\
& - \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} - m_{\mathrm{c}}}
\end{aligned} \tag{7.5}
$$

$$
\begin{aligned}
\frac{i}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} + \mathrm{i}\not{D} - m_{\mathrm{c}}} =& \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} - \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} + \\
& + \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} - \\
& - \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}} i\not{D} \frac{1}{\not{p}_{\mathrm{c}} + \not{p}_{\mathrm{g}} - m_{\mathrm{c}}}
\end{aligned} \tag{7.6}
$$

$$
\frac{i}{\not{p}_{\mathrm{b}} - \not{p}_{\mathrm{g}} + \mathrm{i}\not{D} - m_{\mathrm{b}}} = \frac{1}{\not{p}_{\mathrm{b}} - \not{p}_{\mathrm{g}} - m_{\mathrm{b}}} - \frac{1}{\not{p}_{\mathrm{b}} - \not{p}_{\mathrm{g}} - m_{\mathrm{b}}} i\not{D} \frac{1}{\not{p}_{\mathrm{b}} - \not{p}_{\mathrm{g}} - m_{\mathrm{b}}} + \tag{7.7}
$$

$$+\frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}} i\not{D} \frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}} i\not{D} \frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}} - \tag{7.8}$$
$$-\frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}} i\not{D} \frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}} i\not{D} \frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}} i\not{D} \frac{1}{\not{p}_{\mathrm{b}}-\not{p}_{\mathrm{g}}-m_{\mathrm{b}}}$$

The resulting expressions can be evaluated with the help of the trace formulae 4.34. Due to the fact that more than one consecutive propagator is expanded, traces with more than two iD arise. These term have to be set to zero, as we are computing the $\mathcal{O}(1/m_{\mathrm{b}}^2)$ corrections.

Taking the imaginary part gives different cuts on the c-quark propagators, which turn into delta functions and their derivative depending on the order in iD. Derivatives have to be taken of the delta functions by partial integration. The phase space integration as used for computing the perturbative corrections can be reused here.

Only the real corrections with $(\hat{s}_0 - \rho)^3$ for $\hat{\mu}_\pi^2$ have been computed yet, which requires a statement:

- The corrections for $\hat{\mu}_{\mathrm{G}}^2$ involve gluon matrix elements and as we calculate the radiative corrections a three-gluon vertex, emitting a soft gluon from the radiative gluon, has to be taken into account to preserve gauge invariance. The correct application of the background field formalism in this case has to be worked out. The existing tree-level HQE formalism shifts the b-quark momentum by the residual momentum k, affecting the c-quark propagator through momentum conservation. The c-quark propagator is then expanded in k. In the case of the radiative corrections the momentum of the gluon could be shifted by the momentum k, to obtain a three-gluon vertex as required by gauge invariance. Up to now it is unclear how to apply the HQE in this case. It is probably required to trace the momentum k back to its origin in the background field formalism.
- We have seen in 5.2.2 that the tree-level result and the virtual corrections of the partonic moments with $(\hat{s}_0-\rho)^{>0}$ are zero and thus these corrections can be obtained by computing the real corrections alone. In the case of the radiative corrections to $\hat{\mu}_\pi^2$ the argument changes due to the derivatives of the on-shell delta function for the final c-quark, which is due to the expansion of the c-quark propagator and the application of the cut. The weight of the moment $(\hat{s}_0-\rho)$ is affected by the partial integration for taking the derivative off the delta function. At $\mathcal{O}(1/m_{\mathrm{b}}^2)$ the highest derivative is 2 and thus two partial integrations change the weight yielding a non-zero contribution for the virtual corrections, unless the power of the weight is larger than 2. If this is e. g. $(\hat{s}_0-\rho)^3$ the second derivative due to the partial integration preserves a single power of the weight $(\hat{s}_0 - \rho)$, which turns the virtual contribution of the moment to zero. We can also argue by considering the real corrections only: The real corrections normally are IR divergent. The weight $(\hat{s}_0 - \rho)$ removes the corresponding singularity. The expansion of the propagator yields denominators of higher power, requiring higher powers in the weights to remove the singularity. At $\mathcal{O}(1/m_{\mathrm{b}}^2)$ the power of the denominator is 3 and thus the weight has to be at least $(\hat{s}_0 - \rho)^3$, to yield a finite moment.

In the context of the HQE fit the highest power of $(\hat{s}_0 - \rho)$ needed for the moment analysis is 3 and thus we only compute one single moment of the building blocks shown in table 7.1 which is in agreement with [21].

i	j	$\boldsymbol{H_{ij}}$
3	0	0.05843

Table 7.1: *Coefficient of the H_{30} moment of the radiative correction to $\hat{\mu}_\pi^2/m_b^2$*

8 Summary

8.1 Complete Michel Parameter Analysis of inclusive semileptonic $b \to c$ transition

The calculation of the leptonic moments show that the radiative corrections are as important as the non-perturbative ones. We calculated the complete expressions to order α_s and $1/m_b^2$ for all possible contributions. This is also true for the hadronic moments, but for $i > 0$ they have no tree-level contribution, hence the $\mathcal{O}(\alpha_s)$ and $\mathcal{O}(1/m_\mathrm{b}^2)$ are the leading contributions.

Due to large radiative corrections in the leptonic moments and hadronic moments with $i = 0$ the tree-level and radiative corrections almost cancel out, resulting in a large scale dependence. Thus the sensitivity to scalar and tensor couplings are low and we propose to perform the anticipated combined HQE fit with only the left-handed and right-handed vector couplings, which are known to NLO, due to the vanishing anomalous dimension. The coupling c_R seems to be promising for a moment analysis and may help understand the tension between $|V_{cb}|$ from inclusive decays and $|V_{cb}|$ from exclusive decays.

8.2 Limit on a Right-Handed Admixture to the Weak $b \to c$ Current from Inclusive Semileptonic Decays

We have performed a full-fledged fit to moments of the lepton-energy and hadronic-mass distribution of semileptonic $\bar{B} \to X_c \ell^- \bar{\nu}$ decays, including a possible right-handed admixture to the $b \to c$ current. We have considered the non-standard contributions up to $1/m_b^2$ in the non-perturbative and $\mathcal{O}(\alpha_s)$ in the perturbative corrections. The corresponding fit in the framework of the standard-model yields the most precise determination of $|V_{cb}|$, due to the elaborated theoretical description and the precise measurements of the B factories [39]. Our fit, including a right-handed admixture, is in agreement with the standard model assumption of zero for a right-handed contribution. Unfortunately, the result $c'_\mathrm{R} = 0.04\,^{+0.33}_{-0.47}$ reveals a low sensitivity of the fit to a right-handed contribution, compelling us to state the upper relative admixture limit of 0.9 at 95 % confidence level. The moments of the spectra used in the fit are too similar for right- and left-handed contributions, resulting in the low sensitivity and weak bound of c'_R.

Exclusive decays are competitive in the determination of $|V_{cb}|$, given the precise values for the form factors at the non-recoil point obtained from lattice QCD calculations. The same precise values from lattice QCD calculations can be used to obtain a constraint on c'_R, which is considerably stronger than the one obtained from inclusive decays: c'_R=0.01±0.03. Using this result to determine $|V_{cb}|_\mathrm{incl}$ we can compare the tension between $|V_{cb}|_\mathrm{incl}$ and $|V_{cb}|_\mathrm{excl}$ without a right-handed current and with a right-handed current contribution as from exclusive decays. A right-handed current reduces the tension by about 7 % and its significance from a $2.4\,\sigma$ to a $1.7\,\sigma$ deviation.

8.3 The α_s/m_b^2 Corrections to the Inclusive Decay $\overline{B} \to X_c\ell\overline{\nu}_\ell$

The α_s/m_b^2 corrections to $\overline{B} \to X_c\ell\overline{\nu}_\ell$ are the next step in improving the precistion in the determination of $|V_{cb}|$. The experience with both radiative corrections and non-perturbative corrections tempts us to investigate the radiative corrections to $\hat{\mu}_\pi^2$ and $\hat{\mu}_{\mathrm{G}}^2$. While the corrections to $\hat{\mu}_\pi^2$ have been already calculated [21] and can easily obtained by reparametrization invariance, the corrections to $\hat{\mu}_{\mathrm{G}}^2$ seem to be a complicated calculation. It is expected that they have a larger effect on the moments than the corrections to $\hat{\mu}_\pi^2$. With the presented strategies we have paved a way to compute them. It will be crucial to understand and calculate the three-gluon vertices that appear due to gauge invariance.

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