QCD Aspects of Heavy to Light Currents

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Para Sami,
mi familia y
mis abuelos

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Contents

Chapter 1

Introduction

Since the experiment of Rutherford, the scattering among particles has become one of the basic tools in particle physics to look deeply into Nature. This has basically meant to study shorter and shorter distances, with the aim of finding the fundamental structures that describe all physical phenomena. As soon as it was understood, thanks to the De Broglie relation, that for studying short distances, high energy is required, a big race to reach higher energies started, leading, eventually, to huge particle accelerators such as LEP at CERN in Geneva and the TEVATRON at FERMILAB in Chicago.

Whereas the energy of the devices increased, an overwhelming subatomic world appeared, showing up as a complicated world of new particles. Since the 30's, a large advance has been done toward the understanding of this subnuclear scenario, starting with the formulation of QED and the establishment of the electroweak and the strong interactions and culminating in the formulation of the Standard Model of the elementary particle physics (SM) [1–3]. This model has become the standard theory and is an exceptional tool to describe the phenomenology of high energy physics. In particular, the accuracy of the LEP experiments has allowed us to test this model even at the loop level.

However, there is one sector of the SM one could consider not satisfactory, which is the Higgs sector. The reason for this is that its main ingredient, the neutral Higgs boson, has not been discovered. The Higgs sector is necessary to create masses for all particles. Furthermore, the mixing among different quark families originates also in the Higgs sector. The latter is parametrized by a 3×3 unitary matrix, the so-called Cabibbo-Kobayashi-Maskawa matrix (CKM) [4, 5].

The CKM matrix contains the only source of CP violation within the SM. The effect of CP violation has been observed up to now only in a few systems, namely the neutral K and B mesons. Currently the CKM picture undergoes a detailed experimental test, but until now theoretical predictions and experimental data seem to agree. However, there is still hope to find new effects in this sector, since it is known that the CP violation required for explaining the baryogenesis is larger than the one provided by the SM. In conclusion, it can be stated that the CKM sector of the SM has not been tested at a satisfactory level of accuracy.

In order to test this sector, one has to study flavor changing charged and neutral currents. Within the SM framework, these processes are mediated by electroweak currents involving $W^{\pm}$ and Z^0 bosons and are described in terms of quarks. However, in the experiment only hadrons are observable and, in order to connect SM parameters with the data, one has to deal with the problem of confinement in the strong interaction. This problem has not yet been solved, since it requires a non-perturbative ansatz such as lattice QCD.

Over the last decade significant progress in the description of weak heavy flavor decays has been made. These decays are also subject of large experimental efforts and are about to become one of the best possibilities to test some of the parameters of the CKM matrix. The B factories at SLAC and KEK are producing a lot of information about decays of B-meson with high statistics and low systematic errors. Therefore it is mandatory for theory to try to match the experimental accuracy, i.e. to reduce the uncertainties induced by our poor control over QCD interactions at large distances.

This can be achieved by means of factorization theorems which separate the perturbative physics from the non-perturbative one. There are two kinematical regions where these factorization proofs can be stated at the level of Lagrangian and operators in a model independent way by means of two effective field theories. The first is Heavy Quark Effective Theory (HQET) [6–14], which describes interactions of heavy quarks with soft gluons and quarks, and the second is Soft Collinear Effective Theory (SCET) [15–27], which involves energetic light particles almost on-shell. The main impact of these theories relies on the appearance of symmetries in the leading order Lagrangian that were not present in the full one [28–32].

Corrections to the leading terms can be obtained systematically. The first kind of corrections are parametrized in terms of higher dimensional operators, the dimension of which is compensated by couplings proportional to inverse power of m, the mass of the heavy quark, or E, the energy of the collinear particles. However, this requires to introduce additional unknown non-perturbative parameters or even functions. Secondly, the QCD radiative corrections are governed by the hard scale, which can be obtained perturbatively

Both theories together with Heavy Quark Expansion [33–42] settle the basis to study the QCD interaction of heavy meson weak decays, and the specific subject of this thesis are transitions mediated by heavy to light currents.

Heavy to light currents transition in the limit in which the mass of the heavy quark is large in comparison with the soft momenta carried by the light components can be studied using HQET and in those in which the momenta transfered to the light quark is large with SCET.

The method of the effective field theory corresponds to expanding full QCD operator by an OPE [43, 44], their coefficient are operators with the correspondent dimensionality and quantum numbers. The effective operators describe the low energy physics and are multiplied by Wilson coefficients which factorize the short distance interactions.

In the HQET limit, matrix elements of heavy to light currents operators in the full and in the effective theory will be parametrized in terms of non-perturbative universal form factors, for which no reliable method of calculation exists. However, in the effective theory symmetries appear and relation between them can be established. In particular, matrix elements between the vacuum and B or B^* meson states, which define meson decay constants, are described, at leading order, by only one form factor in the effective theory. Therefore, ratios of meson decay constants, such as f_{B^*}/f_B, are given at leading order by ratios of the corresponding matching coefficients which are computable perturbatively.

Based on an analysis of singularities in the Borel plane, one can obtain the behaviour of the perturbative series for large L, where L is the order of perturbation theory. These singularities yield renormalons ambiguities (see review [45]). The nearest singularity determines the leading asymptotic behaviour.

In schemes without strict separation of large and small momenta, such as $\overline{\text{MS}}$, this procedure artificially introduces infrared renormalon ambiguities in matching coefficients and ultraviolet renormalon ambiguities in HQET matrix elements.

For a physical observable, without renormalons ambiguities, it has been proven that the infrared renormalons in the coefficients, corresponding to information on long distances, cancel against ultraviolet ones in the matrix elements [46, 47], containing information on short distances.

However, this has been shown explicitly only in the large-β_0 limit, which relation to real QCD is unclear. Assuming that this holds beyond this approximation, one may obtain additional information, in a model independent way [48, 49], on the structure of the infrared renormalon singularities of matching coefficients, based on ultraviolet renormalons in higher-dimensional matrix elements, which are controlled by renormalization group methods.

Singularities in the Borel plane are branch points, whose powers are determined by the relevant anomalous dimensions, but normalization factors cannot be calculated. The asymptotic behaviour of the perturbative series for the leading QCD/HQET matching coefficients (due to the nearest infrared renormalon) was studied in [46, 50, 51][1] in the large-β_0 limit. In this thesis, an analysis beyond this approximation is presented [52].

An outstanding progress has been reached in the study of inclusive decays [53–59] using Heavy Quark Expansion. The heavy mass set the large scale and the (differential) rates are expressed by means of an Operator Product Expansion in terms of a series of increasing dimension local operators [60–63]. In the context of differential rates for inclusive decays into light particles in order to extract the matrix element V_{ub} of the CKM matrix experimental cuts on the energy of the outgoing particles are necessary to suppress the large background signal of charm production forcing the kinematics to the end point region of the spectrum , where the final state hadron carries large energies $E_H \sim m_Q$, but small invariant mass $s_H \sim m_Q \Lambda_{QCD}$. In this region, the OPE breaks down, but this problem can be solved by a resummation of certain terms of the OPE. This leads to non-local operators evaluated on the light cone, the matrix elements of which lead to non-perturbative functions, the so-called shape functions [60–62].

A systematic treatment can be performed in the end point region mainly in two steps [16,64,65]. First, matching QCD to SCET, where the hard short distance fluctuations are integrated out and a second matching onto HQET where the collinear degrees of freedom are integrated out after a decoupling of soft and collinear degrees of freedom has been performed by a field redefinition.

[1] Note a typo in (4.8) of [50]: denominators of both terms with a should be 2π, not π.

A study of the leading shape functions, including the radiative corrections, shows that differential decay can be written in terms of $d\Gamma = H \cdot J \otimes S$ [16, 64–66], where H is the hard kernel accounting for the hard fluctuation of order m_Q, J is a Jet function, of the collinear scale $\sqrt{m_Q \Lambda_{QCD}}$, S, is the leading shape function and the symbol $\otimes$ represent a convolution integral.

In order to extract V_{ub} with a good accuracy, a study of power suppress contributions is needed. A tree analysis of them has been performed previously in [63,67–69]. Here, the first step toward a systematic study of the subleading terms using SCET is presented at tree level. New shape functions appear, which have not considered previously. Moreover, it will be showed that the factorization formulae hold beyond leading order [70].

In deriving the Heavy Quark Theory, one introduces the velocity vector v of the hadron in order to split the momentum of the heavy quark into a large $m_Q v$ and a small or "residual" $k \sim \Lambda_{QCD}$ component, $p_Q = m_Q v + k$. As long as the velocity fulfills that the residual momentum $k \sim \Lambda_{QCD}$, one has the freedom to redefine it. This reparametrization freedom is the so-called Reparametrization Invariance (RI) [52, 71–78], its main feature is that connects different orders of the $1/m_Q$ expansion and survives renormalization [74, 75], which means that relations obtained at tree level hold.

In this thesis, the consequences of Reparametrization Invariance for the non-local operators apparent in the endpoint region are studied. The main result is that the number of unknown functions that appear at $1/m_Q$ are reduced [77].

The present thesis is organized as follows:

In Chapter 2 and 3 an introduction of HQET and SCET including corrections to the limiting behaviour and a derivation of the heavy to light currents following [21,22] in the SCET limit is presented.

Chapter 4 is devoted to the study of heavy to light currents for an arbitrary Dirac structure and B-meson decay constants up to $1/m$ at next to leading order.

The asymptotic behaviour of the leading order Wilson coefficients as well as for ratios of meson decay constants will be presented in Chapter 5.

In Chapter 6 power corrections for B-meson inclusive decays will be studied in the end point region of the spectrum using the SCET formalism, a factorization formulae at subleading order will be presented; results are collected in Appendix B. Chapter 7 will be devoted to the study of the consequences to consider RI in the end point region in the HQET framework. Finally, conclusion are exposed in Chapter 8.

Chapter 2

Heavy Quark Effective Theory

Contents

2.1 Effective Theory

Looking at the known particles, one realizes the amazing range covered by the mass spectrum, going from a few eV up to hundreds of GeV. Due to this large variation of mass scales, one common situation in particle physics phenomenology is that a physical problem may involve a widely separated energy scales allowing the study of the low-energy dynamics, independently of the high energy interactions.

The standard examples are the weak decays of leptons and hadrons which involve energy scales around MeV or GeV but are mediated by the gauge boson fields, W and Z, whose masses round the hundreds of GeV.

For such a scenario, a description that does not explicitly involve the heavy gauge boson should be possible. In other words, one expects to switch to a low-energy effective theory [79] in which the heavy degrees of freedom have been removed.

In general, that can be done due to the decoupling theorem [80] which states that a heavy particle decouples from the light ones in the limit in which its mass tends to infinity. The only remnant of the heavy particles will be a logarithm dependence with the mass of the heavy quark absorbed in the coupling constant and a series of inverse power of the heavy quark mass.

This corresponds to an expansion in inverse powers of the heavy quarks mass up to logarithm of the Green functions of the theory. This expansion may be obtained from the Lagrangian field theory, which is called an effective theory [81, 82].

The construction of the effective theory Lagrangian involves several steps [44, 82, 83]. After having identified the massive degrees of freedom, these degrees are removed by integrating it out from the functional integral of the full theory. This is possible since at low energies the heavy particle does not appear as an external state. However, whereas the action of the full theory is usually a local one, what results after this first step is a non-local effective functional action. The non-locality is related to the fact that in the full theory the heavy particle with mass M can appear in virtual processes and propagate over a short but finite distance. The second step is required to obtain a local effective Lagrangian. That will be done by rewriting the non-local effective action as an infinite series of local operators with increasing dimensions by means of an Operator Product Expansion (OPE) [43, 84]. This, roughly speaking, corresponds to an expansion in powers of $1/M$. In this step, the short and long distance physics is disentangled. The long distance physics, corresponding to low energy interactions, is the same in both the full and the effective theory and is described by the operators. The short distances effects arising from quantum corrections involving large virtual momenta are not described correctly in the effective theory since the heavy degrees of freedom have been removed. They appear as a coupling constant of the operators, the so-called Wilson coefficients [43], which are dealt with renormalization-group techniques.

The dimension of the effective Lagrangian is four. The leading term in the expansion generally corresponds to a four-dimension operator which has dimensionless coupling defining the renormalizable piece of the effective Lagrangian. This will be used to calculate the renormalization group flow in the effective theory. The higher-dimensional operators have dimensional coupling constants, the scale of which is set by the heavy mass. However, they do not spoil the renormalizability of the effective theory since up to some given order in the heavy mass expansion only a finite number of higher dimensional operators appears and any finite number of insertions may be renormalized.

There is a slightly different approach to construct the effective theory. Knowing the light degrees of freedom, the symmetries, and the expansion parameter of the system to describe, one can build the effective Lagrangian as a sum over all possible operators compatible with the symmetries of the system up to certain order and involving only the light degrees of freedom. Each operator will have associated a coefficient which will be determined by imposing that a certain energy scale μ, typically of the mass of the heavy particle. Both the Green functions of the full and effective theory have to match. In this way, the coefficient of the operators of the effective theory are determined in terms of the parameters of the full one. These coefficients are the Wilson coefficients accounting for the short distance physics.

The scale μ is introduced to disentangle the short from the long distances, and generally, is chosen such that $\Lambda_{QCD} \ll \mu \ll M$. In this way, the effective theory is derived to be equivalent to the full one at long distances. The short distance effects are absorbed in the coupling constant (Wilson Coefficient). Moreover, the non-dependence of the physical quantities in the scale choice will allow the establishment

of the renormalization group equation.
Finally, remember the advantages of the effective theory over the full one. First, there are examples, where the full theory is not known or the matching is not suitable. In these cases, it is obvious that one performs the calculation in the effective theory. But even in cases where the matching is possible, like in HQET, it may be advantageous to switch to an effective theory mainly for two reasons. First, the renormalization group equation can be used for resumming the large logarithms apparent in perturbation theory for the presence of two disparate scales and second, it may happen that new symmetries appear. That is the case of HQET which will be introduced in some detail in the next section.

2.2 HQET Lagrangian

Heavy Quark Effective Theory is an effective field theory derived from QCD describing physical systems with a heavy quark interacting with the light degrees of freedom mainly by exchange of soft gluons. The high-energy scale is given, in this case, by the heavy quark mass, m_Q and the light degrees of freedom by Λ_{QCD}, a typical energy scale of the hadronic physics.

At the energy scale of the heavy particle the physics is perturbative. Therefore, it can be calculated in QCD. For energies lower than the heavy quark mass the physics becomes nonperturbative and complicated because of confinement. The goal is to find a simplified description in this region using an effective field theory.

In contrast to other effective field theories, HQET has the particularity that the heavy particle does exist as an external particle. Therefore, it will not be possible to remove the heavy quark field completely. However, the heavy degrees of freedom can be identified by means of field redefinitions and consequently, they can be integrated out.

The starting point is the full QCD Lagrangian with a heavy quark [41]:

$$\mathcal{L}(x) = \bar{Q}(x)(i\not{D} - m_Q)Q(x)$$

where $Q(x)$ is the heavy quark field, m_Q is the heavy quark mass and $D^\mu = \partial^\mu + igA^\mu$ is the covariant derivative. Noting that the velocity of the heavy quark inside of an hadron is almost the same as the hadron itself and is almost on-shell, its momentum may be expressed as:

$$p_Q^\mu = m_Q \cdot v^\mu + k^\mu \tag{2.1}$$

where v is the four-vector velocity of the hadron $v = P_{Hadron}/m_{Hadron}$ and therefore $v^2 = 1$. k is the so-called residual momentum being of the order of Λ_{QCD}. Interactions of heavy quarks with the light degrees of freedom, light quarks or soft gluons, change the residual momentum by an amount of the order of $\Delta k \sim \Lambda_{QCD}$, but the corresponding change in the heavy quark velocity vanishes in the limit $\Lambda_{QCD}/m_Q \to 0$. Then, v is conserved and becomes a good quantum number.

The derivation of the HQET Lagrangian from QCD involves a redefinition of the quark fields Q,

$$Q(x) = e^{-im_Q v\cdot x} Q_v(x) \tag{2.2}$$

where the large momentum dependence of the heavy quark field has been explicitly identified. $Q_v(x)$ can be decomposed into:

$$Q_v(x) = h_v(x) + H_v(x) \tag{2.3}$$

where h_v and H_v are, respectively, the small and the large components of the field. The covariant derivative acting over the redefined field is:

$$iD^\mu Q(x) = e^{-im_Q v\cdot x}(m_Q v^\mu + iD^\mu)Q_v(x) \tag{2.4}$$

being $(iD^\mu)Q_v \sim \Lambda_{QCD}$ since corresponds to the residual momentum. Therefore, it follows that $iD^\mu h_v(x) \sim iD^\mu H_v(x) \sim \Lambda_{QCD}$. Moreover, the small and large components can be written in terms of the original field by:

$$\begin{aligned} h_v(x) &= e^{im_Q v\cdot x} P_+ Q(x) \\ H_v(x) &= e^{im_Q v\cdot x} P_- Q(x) \end{aligned}$$

where $P_\pm$ are projection operators defined as:

$$P_\pm = \frac{1 \pm \not{v}}{2} \tag{2.5}$$

The new fields satisfy $\not{v} h_v = h_v$ and $\not{v} H_v = -H_v$. In the rest frame, $v^\mu = (1, \vec{0})$. Then, h_v corresponds to the upper components of the field and annihilates a heavy quark with velocity v whereas H_v corresponds to the lower components and creates a heavy antiquark. Expressing the QCD Lagrangian in terms of the new fields, it can be expressed as:

$$\mathcal{L} = \bar{h}_v(iv\cdot D)h_v - \bar{H}_v\{(iv\cdot D) + 2m_Q\}H_v + \bar{h}_v i\not{D}_\perp H_v + \bar{H}_v i\not{D}_\perp h_v, \tag{2.6}$$

where $D_\perp^\mu = D^\mu - v^\mu(v\cdot D)$ and fulfills $\{\not{D}_\perp, \not{v}\} = 0$. From the Lagrangian expression, it is apparent that h_v is a massless field and therefore describes the light degrees of freedom of the heavy quark. On the other hand, H_v is a field of mass $2m_Q$ and describes the heavy degrees of freedom which will be removed. At the classical level this can be done by solving the equation of motion for the large field H_v,

$$H_v(x) = \left(\frac{1}{2m_Q + iv\cdot D}\right) i\not{D}_\perp h_v(x). \tag{2.7}$$

This shows that the large field is of order $1/m_Q$ and thus, in the heavy mass limit $m_Q \to \infty$, the field vanishes. Inserting this expression in the heavy quark field one gets the non-local effective heavy quark field:

$$Q(x) = e^{-im_Q v\cdot x}\left[1 + \left(\frac{1}{2m_Q + iv\cdot D}\right) i\not{D}_\perp\right] h_v(x) \tag{2.8}$$

In a similar way, for the Lagrangian one gets the non-local effective Lagrangian

$$\mathcal{L}_{HQET} = \bar{h}_v(iv\cdot D)h_v + \bar{h}_v i\not{D}_\perp \left(\frac{1}{2m_Q + iv\cdot D}\right) i\not{D}_\perp h_v \tag{2.9}$$

The non-locality of the Lagrangian is connected with propagating modes of high energy at virtual level [41]. In a virtual level, it is possible that a heavy quark transforms into a heavy antiquark and after a short time, the latter becomes a quark again. The energy of the propagating antiquark is at least $2m_Q$ larger than the energy of the quark and can propagate over a short distance $\Delta x \sim 1/2m_Q$. This kind of processes are generated by the mixing term in (2.6) and corresponds to the interaction

$$T\left(\bar{h}_v i\not{D}_\perp H_v \bar{H}_v i\not{D}_\perp h_v\right) .$$

Contracting the large heavy quark fields, one obtains the propagator recovering, in this way, the non-local term of the effective Lagrangian. The non-local terms can be expanded in terms of an infinite series of local operators. In the framework of the effective theory, this corresponds to a short distance expansion and hence, these operators have to be renormalized.

The tree level relation may be derived from the geometrical series expansion of the non-local terms. The covariant derivative acting over the effective fields is of order $(iv\cdot D)h_v \sim \Lambda_{QCD}$. The mass of the heavy quark sets the heavy scale, then one can make an expansion in terms of powers Λ_{QCD}/m_Q. Hence, the field reads as:

$$\begin{aligned} Q(x) &= e^{-im_Q v\cdot x}\left[1+\left(\frac{1}{2m_Q+iv\cdot D}\right)i\not{D}_\perp\right]h_v = \\ &= e^{-im_Q v\cdot x}\left[1+\frac{1}{2m_Q}i\not{D}_\perp\right]h_v + \mathcal{O}\left(\Lambda^2_{QCD}/m_Q^2\right). \end{aligned} \tag{2.10}$$

In an equivalent way, the Lagrangian is written as:

$$\begin{aligned} \mathcal{L}_{HQET} &= \bar{h}_v\,(iv\cdot D)\,h_v + \bar{h}_v\,i\not{D}_\perp\left(\frac{1}{2m_Q+iv\cdot D}\right)i\not{D}_\perp\,h_v = \\ &= \bar{h}_v\,(iv\cdot D)\,h_v - \frac{1}{2m_Q}\bar{h}_v\,\not{D}_\perp\,\not{D}_\perp\,h_v + \mathcal{O}\left(\Lambda^2_{QCD}/m_Q^2\right) \end{aligned} \tag{2.11}$$

Taking into account that h_v contains a P_+ projection operator, and using the identity

$$P_+\,i\not{D}_\perp\,i\not{D}_\perp P_+ = P_+\left[(iD_\perp)^2+\frac{g_s}{2}\,\sigma_{\mu\nu}\,G^{\mu\nu}\right]P_+\,, \tag{2.12}$$

where $i[D^\mu, D^\nu] = g_s\,G^{\mu\nu}$ is the gluon field-strength tensor, one finds the HQET Lagrangian up to order $1/m_Q$ [11,14]:

$$\begin{aligned} \mathcal{L}_{\rm HQET} = \bar{h}_v\,(iv\cdot D)\,h_v &+ \frac{1}{2m_Q}\,\bar{h}_v\,(iD_\perp)^2\,h_v \\ &+\frac{g_s}{4m_Q}\,\bar{h}_v\,\sigma_{\mu\nu}\,G^{\mu\nu}\,h_v + \mathcal{O}\left(\Lambda^2_{QCD}/m_Q^2\right) . \end{aligned} \tag{2.13}$$

Starting from the generating functional of the Green function of the full QCD, one can get the same effective Lagrangian. The action can be written as [14]:

$$S_{\rm HQET} = \int {\rm d}^4x\,\mathcal{L}_{\rm HQET} - i\ln\Delta\,, \tag{2.14}$$

with $\mathcal{L}_{\text{HQET}}$ given in (2.9). Δ is the determinant coming from the integration of the heavy degrees of freedom and can be ignored.

Finally, one may notice that the effective theory only describes heavy quarks and says nothing about heavy antiquarks. This is due to the choice of the sign in the phase redefinition of the field in order to identify the heavy degrees of freedom. Changing the sign of the global phase in (2.2), one gets a description for heavy antiquarks [38]. In the next section, the symmetries of the effective Lagrangian will be discussed.

2.2.1 Heavy Quark Symmetry

Taking the limit in (2.13) of $m_Q \to \infty$ only the first term remains and yields the effective Lagrangian of the Heavy Quark Effective Theory [12, 85]:

$$\mathcal{L}_\infty = \bar{h}_v \, iv\cdot D \, h_v \, . \tag{2.15}$$

Looking at the Lagrangian one discover two symmetries not present in full QCD. The mass of the heavy quark is absence, since has been removed and there is no any Dirac structure, the soft gluons do not distinguish the spin of the heavy quark. These symmetries are called heavy Flavour and Spin symmetries [28–32]. A direct consequence of the symmetry is the prediction of equal masses for the degenerated states of spins $(0^-, 1^-)$. Moreover, relation between form factors can be found, for example, B meson decay constants in the effective theory are described at leading order by only one form factor. Heavy to heavy transition at leading order are described by the Isgur-Wise form factor [32, 86]. Similar results for exited mesons have been studied in [87]. For D mesons similar results are predicted [88] and confirmed in [89]. For the baryons, similar results have been found [87, 90, 91].

2.3 Corrections to the Heavy Mass Limit

In contrast to model dependent descriptions of QCD, in HQET it is possible to calculate in a systematic way the corrections to the heavy quark limit. These will be given in terms of two small parameters: the strong coupling constant, accounting for the radiative corrections, and Λ_{QCD}/m_Q accounting for the finiteness of the heavy quark mass, the latter being of non-perturbative character.

2.3.1 $1/m_Q$ Correction

Corrections related to the long distances physics are given in terms of a series of increasing dimension operators that scales with inverse powers of the heavy quark mass. These corrections are of non-perturbative character of order Λ_{QCD}. Therefore, at higher order in $1/m_Q$ new parameters or even form factors will appear, which can not be calculated within the HQET framework.

As an example, one can consider a matrix element of a current $\bar{q}\Gamma Q$ mediating a transition between a heavy meson and some arbitrary state $|A\rangle$. For this matrix element, there are two sources of $1/m_Q$ corrections, corrections to the currents and corrections to the states. The corrections to the currents arise from the $1/m_Q$ expansion of the fields (2.8) and will give local contributions

$$Q(x) = e^{-im_Q v\cdot x}\left[1 + \frac{1}{2m_Q} i\not{D}_\perp\right] h_v(x) + \mathcal{O}\left(1/m_Q^2\right) \tag{2.16}$$

whereas the corrections to the states arise from considering the higher order terms of the Lagrangian as perturbative ones and will give rise to non-local contributions. The matrix element under consideration up to order $1/m_Q$ takes the form [1] [92,93]:

$$\begin{aligned}\langle A|\bar{q}\Gamma Q|M(v)\rangle = \langle A|\bar{q}\Gamma h_v|H(v)\rangle + \frac{1}{2m_Q}\langle A|\bar{q}\Gamma P_- i\not{D} h_v|H(v)\rangle \\ -i\int d^4x \langle A|T\{\mathcal{L}_1(x)\bar{q}\Gamma h_v\}|H(v)\rangle + \mathcal{O}(\Lambda_{QCD}^2/m_Q^2)\end{aligned} \tag{2.17}$$

where $\mathcal{L}_1$ are the first-order corrections to the Lagrangian given in (2.13). Furthermore, $|M(v)\rangle$ is the state of the heavy meson in full QCD, including all its mass dependence, while $|H(v)\rangle$ is the corresponding state in the infinite mass limit. When combined with radiative corrections, new operator compatibles with the symmetries of the problem can appear. In Chapter 4, the operator basis at $1/m_Q$ will be given for heavy to light currents in terms of Spin-0 operators, Section 4.3, and Spin-1 operators, Section 4.4.

2.3.2 Radiative Corrections

HQET describes the low energy behaviour of QCD of a heavy quark interacting softly with the light degrees of freedom. Processes involving hard virtual loops are not described properly. This is corrected by Wilson coefficients extracted by comparing both theories a certain scale where perturbative QCD still works.

The Heavy Quark Effective Lagrangian in presence of hard loops is modified to [10,11,13]:

$$\begin{aligned} L &= \bar{h}_v iv\cdot D h_v + \frac{1}{2m_Q}\left[O_k + C_m(\mu)O_m(\mu)\right] + \mathcal{O}(1/m_Q^2)\,, \\ O_k &= -\bar{h}_v D_\perp^2 h_v\,, \quad O_m = \frac{1}{2}\bar{h}_v G_{\alpha\beta}\sigma^{\alpha\beta} h_v\,, \end{aligned} \tag{2.18}$$

O_k and O_m are, respectively, the kinetic and the chromomagnetic operators contain the low energy physics. Due to reparametrization invariance [71], the kinetic-energy operator, O_k, is not renormalized, and its coefficient is unity to all orders in perturbation theory. This will be shown in Sect. 2.5.

The chromomagnetic-interaction coefficient $C_m(\mu)$ can only be found approximately by matching the amplitudes of an appropriate scattering process in QCD and HQET. Its dependence on m_Q is given as a combined expansion in the coupling strength $\alpha_s = g^2/(4\pi)$ with n_l light flavours and logarithms of m_Q,

$$\begin{aligned} C\left(m_Q/\mu\right) &= a_{00} \\ &+ a_{11}\left(\alpha_s \ln\left(m_Q/\mu\right)\right) + a_{10}\alpha_s \\ &+ a_{22}\left(\alpha_s \ln\left(m_Q/\mu\right)\right)^2 + a_{21}\alpha_s\left(\alpha_s \ln\left(m_Q/\mu\right)\right) + a_{20}\alpha_s^2. \end{aligned} \tag{2.19}$$

[1] light quarks are considered to be soft.

This splitting between short and long distances is known as the factorization theorem and corresponds to the statement that the ultraviolet divergences in the effective theory have to match the logarithmic mass dependences of full QCD.

The factorization scale μ is an arbitrary parameter. The Lagrangian does not depend on it. Therefore, differentiating (2.18) with respect to the factorization scale μ yields the renormalization group equation

$$\frac{d}{d\ln\mu}\left\{C_m\left(\frac{m_Q}{\mu}\right)O_m(\mu)\right\}=0 \tag{2.20}$$

where O_m is the renormalized chromomagnetic operator. Then, the running of the Wilson coefficient $C(m_Q/\mu)$ is given by:

$$\left(\frac{d}{d\ln\mu}+\gamma_m(\mu)\right)C\left(\frac{m_Q}{\mu}\right)=0, \tag{2.21}$$

where

$$\gamma_m(\mu)=-\frac{d}{d\ln\mu}\ln(O_m(\mu)).$$

is the anomalous dimension, γ_m, of the chromomagnetic operator O_m, in the $\overline{\mathrm{MS}}$ [11, 93–95]

$$\gamma_m=2C_A\frac{\alpha_s}{4\pi}+\frac{4}{9}C_A(17C_A-13T_Fn_l)\left(\frac{\alpha_s}{4\pi}\right)^2+\cdots \tag{2.22}$$

Equation (2.21) defines the renormalization group equation in the effective theory and allows one to shift the logarithm dependence in the Wilson coefficient. Solving the renormalization-group equation yields

$$C_m(\mu)=\hat{C}_m\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{\gamma_{m0}}{2\beta_0}}K_{-\gamma_m}(\alpha_s(\mu))\,, \tag{2.23}$$

in the $\overline{\mathrm{MS}}$ scheme, $\beta(\alpha_s)=-\frac{1}{2}d\log\alpha_s/d\log\mu=\beta_0\alpha_s/(4\pi)+\beta_1(\alpha_s/(4\pi))^2+\cdots$, and for any anomalous dimension $\gamma(\alpha_s)=\gamma_0\alpha_s/(4\pi)+\gamma_1(\alpha_s/(4\pi))^2+\cdots$ and

$$K_\gamma(\alpha_s)=\exp\int_0^{\alpha_s}\left(\frac{\gamma(\alpha_s)}{2\beta(\alpha_s)}-\frac{\gamma_0}{2\beta_0}\right)\frac{d\alpha_s}{\alpha_s}\,. \tag{2.24}$$

Equation (2.23) sums the large logarithms that appear in (2.19). Calculating the renormalization group functions β and γ at two loops and the finite Wilson coefficient at one loop up to the subleading logarithms ($\alpha_s^{n+1}\ln^n m_Q$) are resummed, corresponding to a resummation of the first and second row of (2.19).

The full one-loop correction to C_m has been calculated in [10], and the two-loop correction in [95]. One obtains(see [96])

$$\hat{C}_m=1+c_{m1}\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\,, \tag{2.25}$$

$$c_{m1}=2C_F+\frac{5}{2}C_A-\left(3C_F+\frac{55}{6}C_A\right)\frac{C_A}{\beta_0}+(11C_F+7C_A)\frac{C_A^2}{\beta_0^2}\,.$$

where

$$\mu_0 = e^{-5/6} m \,. \tag{2.26}$$

In Chapter 4 and 5 radiative corrections of matrix elements of heavy to light currents will be discussed including $1/m$ corrections. These matrix elements, are impossible to calculate from first principles, with the exception of Lattice calculations. However, (2.21) allows the extraction of the short distance piece, i.e., the logarithms of the large mass m and to shift them into the Wilson coefficients.

The radiative corrections will be included in the effective theory following the steps. First, a calculation in both QCD and HQET is performed. After subtracting the ultraviolet divergences the matching is carried out, at the large scale m_Q, getting in this way the Wilson coefficients that account for the short distance behaviour. In this process, a scale which is used for obtaining the renormalization group equation of the effective theory is introduced. This is used for lowering the renormalization point at scales $\mu < m_Q$ where one can switch to the effective theory, performing at the same time the resummation of the problematic large logarithms that appear in the full theory.

2.4 Parameters of the Effective Lagrangian

Looking at the effective Lagrangian, one may notice the presence of two parameters: the velocity v and the mass of the heavy quark. The velocity appears after the splitting of the heavy quark momentum into a large and a residual part breaking the Lorentz invariance, since a reference system is chosen. This can be restored by integrating it in more degrees of freedom [85],

$$\mathcal{L}_\infty = \sum_v \bar{h}_v (iv \cdot D) h_v \tag{2.27}$$

But two heavy quarks with different velocities are separated by a gap of infinite energy $\Delta P = m_Q(v - v')$ with $m_Q \to \infty$ and hence, independent. This is known as a Superselection rule. Then, from now on, only quarks moving with a certain velocity will be considered.

Moreover, the velocity is not present in QCD and from that point of view it can be considered as an external variable. This means that redefinitions of the velocity of compatible size with the dynamical assumptions to derive HQET should leave the physical results invariant. This is the so-called Reparametrization Invariance [71, 72, 74–76] explained in the next section. Its application to the end point region will be studied in Chapter 7.

The other parameter that appears in the Lagrangian is the mass of the heavy quark. The mass of the heavy particle is not a physical quantity, because of confinement, and one has some freedom to define it. Redefinition of the mass $m_Q = m_Q + \delta m_Q$ amount to redefinitions of the so-called binding energy of the hadron H_Q:

$$\bar{\Lambda} = (m_{H_Q} - m_Q)\Big|_{m_Q \to \infty} \,. \tag{2.28}$$

Physical quantities are independent of this choice [97]. For $\delta m = 0$, as it is usually implicitly done in the HQET, a heavy-quark mass is defined, the pole mass of the heavy quark [98].

Finally, one may note that m_Q, $\bar{\Lambda}$ and δm are non-perturbative quantities, although, they are evaluated by means of perturbative methods. A careful study of the perturbative series reveals that it not possible to define these quantities unambiguously [99, 100]. The perturbative expansion shows a factorial divergence for large orders, which is related with the appearance of singularities in the right side of the real axis of the Borel plane, the so-called Renormalons [45, 48, 101–109]. There is no unique way to deal with these singularities and hence, an ambiguity appears. As far as physical quantities are concerned all ambiguities cancel [46, 47, 110]. This has been proof for many applications in the large-β_0 limit. In Chapter 5, it will be shown this cancellation by a direct calculation in the large-β_0 limit for heavy to light currents in the HQET framework.

2.5 Reparametrization Invariance

In deriving the heavy mass expansion from QCD, one introduces a velocity vector v which is the velocity of the hadron containing the heavy quark, $v = P_{hadron}/M_{hadron}$. The heavy quark momentum p_Q inside the heavy meson is decomposed into a large part $m_Q v$ and a residual part k,

$$p_Q = m_Q v + k$$

v must be chosen such that the residual momentum $k \sim \Lambda_{QCD}$. Then, the heavy mass expansion is constructed by expanding the amplitudes in the small quantity k/m_Q. However, the splitting of the momentum is not unique, a different choice

$$p_Q = m_Q v' + k'$$

is valid (as long as $v'^2 = 1$) such that $v' = v + \Delta$ and $k' = k - m_Q \Delta$, with $\Delta \sim \Lambda_{QCD}/m_Q$ to maintain $k' \sim \Lambda_{QCD}$. From this point of view, the velocity vector v in HQET is an external variable which is not present in full QCD and is only fixed up to terms of the order Λ_{QCD}/m_Q. Consequently, small reparametrizations of the form $v \to v + \Delta$ with $\Delta = \mathcal{O}(\Lambda_{QCD}/m_Q)$ should leave the physical results of the heavy mass expansion invariant.

This so-called Reparametrization Invariance(RI) is known since the early days of heavy quark effective theory (HQET) [71, 72]. Like other symmetries RI will be useful for constraining the shape of the Lagrangian and physical observables. The HQET Lagrangian, like the full one, is invariant under small changes of the velocity as long as all the terms of the $1/m_Q$ expansion are considered. In other words, the truncation of the expansion up to certain order implies the introduction of a dependence in the velocity up to the same order. The key point of RI is that connects different orders of the $1/m_Q$ expansion. Moreover, RI survives renormalization [74, 75], which means that relations obtained at tree level hold. Therefore, relations among the coefficients of the operators will be obtained, the most prominent of the example is the non-renormalization of the kinetic energy operator $\bar{h}_v (iD)^2 h_v$ [71].

If one considers two versions of HQET with two different choices of the velocity vectors v and v' differing by a small quantity Δ [72], then

$$v^2 = 1; \qquad v'^2 = 1 = (v + \Delta)^2 = 1 + 2v \cdot \Delta + \mathcal{O}(\Delta^2) \tag{2.29}$$

with $v \cdot \Delta = 0$ and such that Δ is of order Λ_{QCD}/m_Q. Then, the two versions of HQET must be equivalent.

Constructing HQET from QCD involves a redefinition of the quark field Q of the form

$$Q = \exp(-im_Q v \cdot x) Q_v \tag{2.30}$$

such that the covariant derivative acts as

$$iD_\mu Q = \exp(-im_Q v \cdot x)(m_Q v_\mu + iD_\mu) Q_v \tag{2.31}$$

The left hand side corresponds to the full heavy quark momentum which is not changed under reparametrization. This implies for the change δ_R of the covariant derivative acting on the quark field Q_v

$$\delta_R(iD_\mu) = -m_Q \Delta_\mu\,. \tag{2.32}$$

In the following, a consistent scheme to count the powers of Λ_{QCD}/m_b in order to expand the Lagrangian in a systematic way has to be developed. Defining the action to be $\mathcal{O}(1)$, one obtains that the static heavy quark field is $\mathcal{O}(\Lambda_{QCD}^{3/2})$. The covariant derivative as well as the variation δ_R of the covariant derivative are $\mathcal{O}(\Lambda_{QCD})$, and the variation of the heavy quark field under reparametrization is

$$\delta_R h_v = \frac{\not{\Delta}}{2}\left[1 + \frac{i\not{D}}{2m_Q}\right] h_v + \mathcal{O}[\Lambda_{QCD}^{3/2}(\Lambda_{QCD}/m_Q)^3]\,. \tag{2.33}$$

Note that the leading contribution originates from the variation of the projector $P_+ = (\not{v} + 1)/2$ and is of order $\Lambda_{QCD}^{5/2}/m_Q$

Equations (2.29), (2.32) and (2.33) are the reparametrization transformations of all relevant quantities needed to exploit the consequences of this symmetry.

Reparametrization invariance connects terms of different orders in the $1/m_Q$ expansion. As an example, the HQET Lagrangian is considered

$$\begin{aligned}\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_1 + \cdots &= \bar{h}_v (iv \cdot D) h_v \\ &+ \frac{1}{2m_Q}\bar{h}_v (iD)^2 h_v - \frac{i}{2m_Q}\bar{h}_v (iD_\mu)(iD_\nu)\sigma^{\mu\nu} h_v + \mathcal{O}(\Lambda_{QCD}^6/m^2)\end{aligned} \tag{2.34}$$

with $h_v = P_+ h_v$ where $P_+ = (1 + \not{v})/2$.

The leading order term $\mathcal{L}_0$ is of order Λ_{QCD}^4, while its variation is of order Λ_{QCD}^5/m_Q

$$\delta_R \mathcal{L}_0 = \bar{h}_v (i\Delta \cdot D) h_v + \mathcal{O}[\Lambda_{QCD}^6/m_Q^2] \tag{2.35}$$

Note that the leading term of the variation of the fields (2.33) does not contribute since

$$P_+ \not{\Delta} P_+ = P_+ (v \cdot \Delta) = 0 \tag{2.36}$$

The variation of the leading-order term is compensated by the kinetic energy term, since

$$\delta_R \left(\bar{h}_v (iv \cdot D) h_v + \frac{1}{2m_Q} \bar{h}_v (iD)^2 h_v \right) = \mathcal{O}[\Lambda^6_{QCD}/m_Q^2] \tag{2.37}$$

Relation (2.37) is preserved under renormalization which ensures that the kinetic energy piece is not renormalized [71].

In a similar way one can obtain relations between higher order terms in the Lagrangian and also for matrix elements. Again these relations do not change under renormalization from which relations between renormalization constants can be derived.

Next Chapter is devoted to Soft Collinear Effective Theory(SCET).

Chapter 3

SCET

Contents

3.1 Introduction

Soft Collinear Effective Theory (SCET) [15–27] is an effective theory derived from QCD describing a jet of light particles almost on-shell moving along one direction. This kind of jet can be produced in a heavy hadron weak decay. SCET together with HQET settle the basis to study in a systematic way heavy quark decays into a jet of light particles. The aim of the effective theory is to expand physical quantities in a small parameter λ. Considering a weak decay of a heavy quark with mass m, the jet of the decaying collinear particle is defined to move along the direction n_- with a momentum of order m to amount an invariant mass of the order $(m\lambda^2)$. SCET deals with the physics below this scale. In this Chapter, the discussion given in [21, 22] is closely followed. The momentum can be decomposed as:

$$p^\mu = (n_- p)\frac{n_+^\mu}{2} + (n_+ p)\,\frac{n_-^\mu}{2} + p_\perp^\mu = p_+\frac{n_+^\mu}{2} + p_-\frac{n_-^\mu}{2} + p_\perp^\mu \equiv (p_+, p_-, p_\perp), \qquad (3.1)$$

where n_- and n_+ are two light cone vector $n_\pm^2 = 0$ and $n_- \cdot n_+ = 2$. The momentum is chosen to scale as:

$$p_+ \sim m\lambda^2, \quad p_- \sim m, \quad p_\perp \sim m\lambda, \qquad (3.2)$$

such that $p^2 \sim m^2\lambda^2$ is fulfilled. From now on, the mass of the heavy quark is set to 1, $m = 1$. Then, the dimension of any quantity is restored by inserting the appropriate power of m. The invariant mass of the jet does not change if a particle with soft momenta λ^2 is added. Therefore, the effective theory has to include these soft degrees of freedom. However, the effective theory does not contain particles with momenta scaling as λ, hard-soft, since adding that to a collinear particles with momentum p implies an invariant mass $(p+k)^2 \sim \lambda$ in contradiction with our kinematical assumption. The Theory is constructed in order to reproduce the Green function of QCD under the above kinematical assumption as an expansion of λ. Hard modes of order m and hard-soft modes of order λ can be integrated out. Effective fields for soft and collinear quarks and gluons will be introduced creating or annihilating soft and collinear (anti-)particles respectively. The collinear field will be split in a large and small component:

$$\psi_c(x) = \xi(x) + \eta(x), \qquad \xi(x) \equiv \frac{\not{n}_-\not{n}_+}{4}\,\psi_c(x), \quad \eta(x) \equiv \frac{\not{n}_+\not{n}_-}{4}\,\psi_c(x), \tag{3.3}$$

where $(\not{n}_\mp\not{n}_\pm)/4$ are projection operators, and $\not{n}_-\xi = \not{n}_+\eta = 0$. In order to construct a systematic expansion, one has to assign a scaling rule for the fields which is obtained from the projection of the QCD propagator. For the ξ fields:

$$\langle 0|T\xi(x)\bar{\xi}(y)|0\rangle = \frac{\not{n}_-\not{n}_+}{4}\,\langle 0|T\psi_c(x)\bar{\psi}_c(y)|0\rangle\,\frac{\not{n}_+\not{n}_-}{4} = \int \frac{d^4p}{(2\pi)^4}\,\frac{\not{n}_-}{2}\,\frac{in_+p}{p^2+i\epsilon}\,e^{-ip(x-y)}. \tag{3.4}$$

dp^4 is of order λ^4, therefore the right-hand side scales as λ^2. From there, and an analogous equation for η, one arrives at:

$$\xi \sim \lambda, \qquad \eta \sim \lambda^2. \tag{3.5}$$

The small component η will be integrated out. For soft fields, q taking into account that $dp_s^4 \sim \lambda^8$, since $p_s^2 = \lambda^4$. One obtains:

$$q \sim \lambda^3. \tag{3.6}$$

If these particles are produced from a heavy quark decay, the heavy quark field has to be included. The effective quark field h_v carries soft momenta, hence, it will scale as $h_v \sim \lambda^3$. Finally, with a similar arguments, the collinear gluons scales as:

$$n_+A_c \sim 1, \quad A_{\perp c} \sim \lambda, \quad n_-A_c \sim \lambda^2. \tag{3.7}$$

For a soft gluon field,

$$A_{\rm s}^\mu \sim \lambda^2. \tag{3.8}$$

Derivatives of fields correspond to their momenta. Therefore, a full covariant derivative scales as,

$$D_s^\mu \phi_{\rm s} = (i\partial + gA_s^\mu)\phi \sim \lambda^2\phi_{\rm s}. \tag{3.9}$$

for a soft field and

$$n_+D_c\phi_c \sim \phi_c, \quad D_c^\perp\phi_c \sim \lambda\phi_c, \quad n_-D\phi_c \sim \lambda^2\phi_c. \tag{3.10}$$

for a collinear field with $iD_c^\mu = i\partial^\mu + gA_c^\mu$. Finally, the integration element d^4x in presence of a collinear and soft fields scale as the inverse of the integration element of a collinear momenta $1/\lambda^4$, only when soft fields appear as $1/\lambda^8$. This is related with the fact that the integration over x eliminates a momentum integral when expressing the fields in their Fourier transform and therefore, eliminating the soft integration in an integral such as $\int d^4p_{1c}\, d^4p_{2c}\, d^4p_{3s}$ would no longer ensure that the momentum $p_3 = -(p_{1c} + p_{2c})$ is soft. In next section the Effective Lagrangian will be derived.

3.2 Effective Lagrangian

First, the Lagrangian that describes only collinear quarks is derived. The starting point is the QCD Lagrangian for massless particles:

$$\mathcal{L}_c = \bar{\psi}_c\, (i\not{D})\, \psi_c, \tag{3.11}$$

where ψ_c is assumed to describe a nearly on-shell particle with large momenta in the n_- direction and $iD = i\partial + gA_s + gA_c$. Decomposing the collinear field as in (3.3) the Lagrangian reads as

$$\mathcal{L}_c = \bar{\xi}\, \frac{\not{n}_+}{2}\, in_- D\, \xi + \bar{\eta}\, \frac{\not{n}_-}{2}\, in_+ D\, \eta + \bar{\xi}\, (i\not{D}_\perp)\, \eta + \bar{\eta}\, (i\not{D}_\perp)\, \xi. \tag{3.12}$$

Solving the equation of motion for the η field,

$$\eta(x) = -\frac{\not{n}_+}{2}\, (in_+ D + i\epsilon)^{-1}\, i\not{D}_\perp\, \xi(x). \tag{3.13}$$

where an appropiate $i\epsilon$ prescription is introduced. From this equation, one confirms that the η fields scales as λ^2. Inserting the small field into the Lagrangian, one gets the non-local effective Lagrangian:

$$\mathcal{L}_c = \bar{\xi}\, in_- D\, \frac{\not{n}_+}{2}\, \xi + \bar{\xi}\, i\not{D}_\perp \frac{1}{in_+ D + i\epsilon}\, i\not{D}_\perp \frac{\not{n}_+}{2}\, \xi \tag{3.14}$$

The non-locality can be shown explicitly by using the properties

$$in_+ D\, W = W\, in_+ \partial, \qquad (in_+ D + i\epsilon)^{-1} = W\, (in_+ \partial + i\epsilon)^{-1}\, W^\dagger. \tag{3.15}$$

with $W(x)$, a Wilson line defined by:

$$W(x) = P \exp\left(ig \int_{-\infty}^{0} ds\, n_+ A(x + sn_+) \right), \tag{3.16}$$

with $WW^\dagger = W^\dagger W = 1$. Now, defining the action of the inverse of $in_+\partial$ with a $+i\epsilon$-prescription as

$$\frac{1}{in_+ \partial + i\epsilon}\, \phi(x) = -i \int_{-\infty}^{0} ds\, \phi(x + sn_+). \tag{3.17}$$

The collinear Lagrangian in a manifest non-local way is written by:

$$\mathcal{L}_c = \bar{\xi}(x)\, in_- D\, \frac{\not{n}_+}{2}\, \xi(x) + i \int_{-\infty}^{0} ds \left[\bar{\xi}\, i\overleftarrow{\not{D}}_\perp W \right](x) \left[W^\dagger\, i\not{D}_\perp \frac{\not{n}_+}{2}\, \xi \right](x + sn_+), \tag{3.18}$$

Before the soft quark are introduced, it is worthwhile to study the gauge symmetry of the theory. The effective theory must have a remnant gauge symmetry from gauge functions $U(x)$ that can themselves be classified as collinear or soft. A soft field with small momentum moves along long distances and therefore, cannot resolve the short distance variations of the collinear fields. On the other side, the collinear fields see the soft field as a background field. Thus A_c transforms covariantly under soft gauge transformations and inhomogeneously under collinear gauge transformations, but with the derivative replaced by the covariant derivative with respect to the background field. These properties are summarized by:

$$\begin{aligned} &\text{collinear:} && A_c \to U_c\, A_c\, U_c^\dagger + \frac{i}{g} U_c \left[D_\mathrm{s}, U_c^\dagger\right], && \xi \to U_c\, \xi, \\ & && A_\mathrm{s} \to A_\mathrm{s}, && q \to q, \\ &\text{soft:} && A_c \to U_\mathrm{s}\, A_c\, U_\mathrm{s}^\dagger, && \xi \to U_\mathrm{s}\, \xi, \\ & && A_\mathrm{s} \to U_\mathrm{s}\, A_\mathrm{s}\, U_\mathrm{s}^\dagger + \frac{i}{g} U_\mathrm{s} \left[\partial, U_\mathrm{s}^\dagger\right], && q \to U_\mathrm{s}\, q, \end{aligned} \tag{3.19}$$

The heavy quark field h_v has transformation properties identical to q. Note that the sum $A_c + A_\mathrm{s}$ transforms in the standard way under both types of gauge transformations. However, $\psi_c = \xi + \eta$ plus q do not transform as the full QCD ψ. The relation of the effective field with the full one is not linear. This is related with the fact of applying the equation of motion for the collinear field. In [22], it has been prooven that the relation of the effective fields with the full QCD one is given by:

$$A = A_c + A_\mathrm{s}, \tag{3.20}$$

$$\psi = \xi + WZ^\dagger q - \frac{1}{in_+D}\frac{\not{n}_+}{2}\left(i\not{D}_\perp \xi + \left[\left[i\not{D}_\perp WZ^\dagger\right]\right] q\right) \tag{3.21}$$

Z is the soft Wilson line define as (3.16) but with the full gauge field replaced by the soft one. It is invariant under collinear gauge and under the soft one transforms as:

$$Z \to U_\mathrm{us}\, Z. \tag{3.22}$$

while W transform as:

$$W \to U_c\, W, \qquad W \to U_\mathrm{s}\, W. \tag{3.23}$$

The "double-brackets" are defined by

$$[[f(D)A]] \equiv f(D)A - Af(D_\mathrm{s}), \qquad [[Af(D)]] \equiv Af(D) - f(D_\mathrm{s})A, \tag{3.24}$$

Soft fields transform homogeneously under gauge transformations, but collinear fields do not. Under collinear gauge transformations, they receive contributions of order λ^2 due to the soft derivative, besides soft field are multiple expanded in presence of collinear fields:

$$\begin{aligned} \phi_\mathrm{s}(x) = {}& \phi_\mathrm{s}(x_-) + \left[x_\perp \partial \phi_\mathrm{s}\right](x_-) \\ & + \frac{1}{2} n_- x \left[n_+ \partial\, \phi_\mathrm{s}\right](x_-) + \frac{1}{2}\left[x_{\mu\perp} x_{\nu\perp} \partial^\mu \partial^\nu \phi_\mathrm{s}\right](x_-) + \mathcal{O}(\lambda^3 \phi_\mathrm{s}). \end{aligned} \tag{3.25}$$

all derivatives on soft fields scale as λ^2. The collinear field multiplying this expansion varies over distances $x_\perp \sim 1/\lambda$ and $n_- x \sim 1$, and therefore, the term with $\partial_\perp \phi_{\rm s}$ is of relative order λ and the terms with $n_+\partial\phi_{\rm s}$ and $\partial_\perp\partial_\perp\phi_{\rm s}$ are of relative order λ^2. In momentum space this Taylor expansion corresponds to the fact that in $p_s + p_c$ the soft momenta along the perpendicular and n_+ direction is small in comparison with the collinear one and therefore are expanded in these small momentum components causing the non conservation of the momenta at the vertex interactions. In coordinate space this corresponds to a breaking of manifest translation invariance due to the Taylor-expansion of soft fields around an arbitrary point in the transverse and n_+ direction which is restored order by order in λ. In the same way, the collinear fields under a soft gauge transformation transform as

$$\begin{aligned} \xi(x) &\to U_{\rm s}(x)\,\xi(x) = U_{\rm s}(x_-)\,\xi(x) + [(x_\perp\partial)\,U_{\rm s}](x_-)\,\xi(x) + \ldots, \\ A_c(x) &\to U_{\rm s}(x_-)\,A_c(x)\,U_{\rm s}^\dagger(x_-) + [(x_\perp\partial)\,U_{\rm s}](x_-)\,A_c(x)\,U_{\rm s}^\dagger(x_-) \\ &\qquad + U_{\rm s}(x_-)\,A_c(x)\,[(x_\perp\partial)\,U_{\rm s}^\dagger](x_-) + \ldots, \end{aligned} \tag{3.26}$$

which clearly shows the in-homogeneity of the transformation law. As a conclusion, the gauge transformations (3.19) mix terms of different order, and hence when replacing the full QCD field in the Lagrangian for the effectives one and expanding in λ. The individual terms $\mathcal{L}^{(i)}$ of the effective Lagrangian (3.38) are no longer invariant under soft and collinear gauge transformations, so that, only the sum $\sum_{i=0}^n \mathcal{L}^{(i)}$ is gauge-invariant up to higher-order corrections.

In order to obtain gauge invariant terms, one needs to find new collinear fields $\hat{\xi}$ and $\hat{A}_c$, such that the Lagrangian expressed in terms of the new field variables is invariant under the homogenized version of the gauge symmetries given by

$$\begin{aligned} &\text{collinear:} \\ &\quad n_+\hat{A}_c \to U_c\, n_+\hat{A}_c\, U_c^\dagger + \frac{i}{g}\, U_c\left[n_+\partial, U_c^\dagger\right], && \hat{\xi} \to U_c\,\hat{\xi}, \\ &\quad \hat{A}_{\perp c} \to U_c\,\hat{A}_{\perp c}\,U_c^\dagger + \frac{i}{g}\,U_c\left[\partial_\perp, U_c^\dagger\right], \\ &\quad n_-\hat{A}_c \to U_c\,n_-\hat{A}_c\,U_c^\dagger + \frac{i}{g}\,U_c\left[n_- D_{\rm s}(x_-), U_c^\dagger\right], \\ &\quad A_{\rm s} \to A_{\rm s}, && q \to q, \\ &\text{soft:} \\ &\quad \hat{A}_c \to U_{\rm s}(x_-)\,\hat{A}_c\,U_{\rm s}^\dagger(x_-), && \hat{\xi} \to U_{\rm s}(x_-)\,\hat{\xi}, \\ &\quad A_{\rm s} \to U_{\rm s}\,A_{\rm s}\,U_{\rm s}^\dagger + \frac{i}{g}\,U_{\rm s}\left[\partial, U_{\rm s}^\dagger\right], && q \to U_{\rm s}\,q. \end{aligned} \tag{3.27}$$

Now every term has the same scaling in λ. Since the soft fields transform homogeneously, no redefinition of these fields is needed. In (3.27) fields and gauge transformations without argument are taken at x as in (3.19), while other arguments are given explicitly. The collinear Wilson line

$$W_c(x) \equiv P\exp\left(ig\int_{-\infty}^{0} ds\, n_+\hat{A}_c(x+sn_+)\right) \tag{3.28}$$

transforms as

$$W_c \to U_c W_c, \qquad W_c \to U_{\rm s}(x_-) W_c U_{\rm s}^\dagger(x_-), \tag{3.29}$$

because the arguments of collinear fields in the path-ordered product correspond to the *same* $(x+sn_+)_- = x_-$. The other objects with simple transformation properties under (3.27) are $\hat{\xi}$, q, $F_{\rm s}^{\mu\nu}$, $in_+\hat{D}_c$, $i\hat{D}_{\perp c}$, $in_-\hat{D}$ (but not $in_-\hat{D}_c$) and $iD_{\rm s}^\mu$. (The "hat" indicates that the covariant derivative contains $\hat{A}_c$, not A_c.) The multipole-expanded Lagrangian will be composed of these objects.

In [22], the relation between the hat fields and the original ones has been found. It is given by:

$$\begin{aligned}
\xi &= R W_c^\dagger \hat{\xi}, \\
gA_{\perp c} &= R\left(W_c^\dagger i\hat{D}_{\perp c} W_c - i\partial_\perp\right) R^\dagger, \\
gn_-A_c &= R\left(W_c^\dagger in_-\hat{D} W_c - in_-D_{\rm s}(x_-)\right) R^\dagger.
\end{aligned} \tag{3.30}$$

with R

$$R(x) = P\exp\left(ig\int_C dy_\mu A_{\rm s}^\mu(y)\right) \tag{3.31}$$

with C a straight path from x_- to x. Here the fields without hats on the left-hand side are in light-cone gauge $n_+A_c = 0$. It is simple to check that the new fields have the required collinear and soft transformations (3.27). Now, all the ingredients in order to obtained the multiple-expanded Lagrangian are settled. This will be done in the next section.

3.3 The Multipole-Expanded Quark Lagrangian

In order to get the effective Lagrangian, first, the full QCD field expressed in terms of the effective one (3.21) is inserted in the full QCD Lagrangian and, second, the field redefinition for the collinear fields (3.30) is performed taken in the collinear light-cone gauge (where $WZ^\dagger = 1$). The resulting expression is expanded in λ. An example of the sort of terms that arise, after the field redefinition, is given by the collinear quark Lagrangian (3.14) which takes the form:

$$\begin{aligned}
\mathcal{L} = {}& \bar{\xi}\, in_-D\frac{\not{n}_+}{2}\xi + \bar{\xi}\, W_c\left(R^\dagger in_-D_{\rm s}(x)R - in_-D_{\rm s}\right) W_c^\dagger \frac{\not{n}_+}{2}\xi \\
& + \bar{\xi}\,\left(i\not{D}_{\perp c} + W_c\left(R^\dagger i\not{D}_{\perp{\rm s}}(x)R - i\not{\partial}_\perp\right)W_c^\dagger\right) W_c R^\dagger \frac{1}{in_+D_{\rm s}(x)} R W_c^\dagger \\
& \left(i\not{D}_{\perp c} + W_c\left(R^\dagger i\not{D}_{\perp{\rm s}}(x)R - i\not{\partial}_\perp\right)W_c^\dagger\right)\frac{\not{n}_+}{2}\xi
\end{aligned} \tag{3.32}$$

where the hat of the fields have been dropped from here on. The in_-D contains the collinear gauge fields at x and the soft gauge field at x_- after multiple expanded. This convention will be taken for all soft fields in presence of collinear ones. Soft fields with derivatives will set to x_- after the derivative applies. From this and similar manipulations of the terms in the Lagrangian with the soft quark field the terms $(R^\dagger in_-D_{\rm s}(x)R - in_-D_{\rm s})$, $(R^\dagger i\not{D}_{\perp{\rm s}}(x)R - i\not{\partial}_\perp)$, $R^\dagger(in_+D_{\rm s}(x))^{-1}R$, $R^\dagger q(x)$ and

$i\not{\partial}_\perp R^\dagger q(x)$ appear which have to be multiple expanded in power of λ. The expansion is performed as [22].

$$R^\dagger in_- D_{\rm s}(x) R - in_- D_{\rm s} = \int_0^1 ds\,(x-x_-)^\mu n_-^\nu\, R^\dagger(y(s))\, gF^{\rm s}_{\mu\nu}(y(s))\, R(y(s))$$
$$= x_\perp^\mu n_-^\nu gF^{\rm s}_{\mu\nu} + \frac{1}{2}\, n_- x\, n_+^\mu n_-^\nu gF^{\rm s}_{\mu\nu} + \frac{1}{2}\, x_\perp^\mu x_{\perp\rho} n_-^\nu [D_{\rm s}^\rho, gF^{\rm s}_{\mu\nu}] + \mathcal{O}(\lambda^5), \qquad (3.33)$$

$$R^\dagger i\not{D}_{\perp{\rm s}}(x) R - i\not{\partial}_\perp = \int_0^1 ds\, s\,(x-x_-)^\mu \gamma_\perp^\nu\, R^\dagger(y(s))\, gF^{\rm s}_{\mu\nu}(y(s))\, R(y(s))$$
$$= \frac{1}{2}\, x_\perp^\mu \gamma_\perp^\nu gF^{\rm s}_{\mu\nu} + \mathcal{O}(\lambda^4), \qquad (3.34)$$

$$R^\dagger \frac{1}{in_+ D_{\rm s}(x)} R = \frac{1}{in_+\partial} - \frac{1}{in_+\partial}\, \frac{1}{2}\, x_\perp^\mu n_+^\nu\, gF^{\rm s}_{\mu\nu}\, \frac{1}{in_+\partial} + \mathcal{O}(\lambda^4), \qquad (3.35)$$

$$R^\dagger q(x) = \sum_{n=0}^\infty \frac{1}{n!}\,(x-x_-)_{\rho_1}\dots(x-x_-)_{\rho_n} D_{\rm s}^{\rho_1}\dots D_{\rm s}^{\rho_n} q$$
$$= q + x_{\perp\mu} D_{\rm s}^\mu q + \mathcal{O}(\lambda^2 q), \qquad (3.36)$$

$$i\not{\partial}_\perp R^\dagger q(x) = \sum_{n=0}^\infty \frac{1}{n!}\,(x-x_-)_{\rho_1}\dots(x-x_-)_{\rho_n} i\not{D}_{\perp{\rm s}} D_{\rm s}^{\rho_1}\dots D_{\rm s}^{\rho_n} q, \qquad (3.37)$$

After this expansion every single term has a homogeneous scaling behavior in λ. With these results, it is easy to write down the multipole-expanded SCET Lagrangian to any order in λ. To order λ^2 the result takes the form

$$\mathcal{L} = \bar\xi \left(in_- D + i\not{D}_{\perp c} \frac{1}{in_+ D_c}\, i\not{D}_{\perp c} \right) \frac{\not{n}_+}{2}\, \xi + \bar q(x)\, i\not{D}_{\rm s}(x) q(x) + \mathcal{L}_\xi^{(1)} + \mathcal{L}_\xi^{(2)} + \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\xi q}^{(2)}, \qquad (3.38)$$

where the power-suppressed interaction terms are given by

$$\mathcal{L}_\xi^{(1)} = \bar\xi \left(x_\perp^\mu n_-^\nu\, W_c\, gF^{\rm s}_{\mu\nu} W_c^\dagger \right) \frac{\not{n}_+}{2}\, \xi, \qquad (3.39)$$

$$\mathcal{L}_\xi^{(2)} = \frac{1}{2}\, \bar\xi \left((n_- x)\, n_+^\mu n_-^\nu\, W_c\, gF^{\rm s}_{\mu\nu} W_c^\dagger + x_\perp^\mu x_{\perp\rho} n_-^\nu W_c [D_{\rm s}^\rho, gF^{\rm s}_{\mu\nu}] W_c^\dagger \right) \frac{\not{n}_+}{2}\, \xi \qquad (3.40)$$
$$+ \frac{1}{2}\, \bar\xi \left(i\not{D}_{\perp c}\, \frac{1}{in_+ D_c}\, x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF^{\rm s}_{\mu\nu} W_c^\dagger + x_\perp^\mu \gamma_\perp^\nu\, W_c\, gF^{\rm s}_{\mu\nu} W_c^\dagger\, \frac{1}{in_+ D_c}\, i\not{D}_{\perp c} \right) \frac{\not{n}_+}{2}\, \xi,$$

$$\mathcal{L}_{\xi q}^{(1)} = \bar q\, W_c^\dagger i\not{D}_{\perp c}\, \xi - \bar\xi\, i\overleftarrow{\not{D}}_{\perp c} W_c\, q, \qquad (3.41)$$

$$\mathcal{L}_{\xi q}^{(2)} = \bar q\, W_c^\dagger \left(in_- D + i\not{D}_{\perp c}\, (in_+ D_c)^{-1}\, i\not{D}_{\perp c} \right) \frac{\not{n}_+}{2}\, \xi + \bar q\, x_{\perp\mu} \overleftarrow{D}_{\rm s}^\mu W_c^\dagger\, i\not{D}_{\perp c} \xi \qquad (3.42)$$
$$- \bar\xi\, \frac{\not{n}_+}{2} \left(in_- \overleftarrow{D} + i\overleftarrow{\not{D}}_{\perp c} \left(in_+ \overleftarrow{D}_c \right)^{-1} i\overleftarrow{\not{D}}_{\perp c} \right) W_c\, q - \bar\xi\, i\overleftarrow{\not{D}}_{\perp c} W_c\, x_{\perp\mu} D_{\rm s}^\mu q. \qquad (3.43)$$

3.3.1 No Renormalization

The effective theory is constructed to describe modes below the scale $(m\lambda^2)$. Therefore, one can think that hard corrections above this scale enter in the theory by radiative corrections generating beyond tree level new operators. In SCET the hard scale is carried along the n_+ direction in $(pn_-)n_+^\mu \sim \lambda^0$. Lorentz invariant object generated in loop at the hard scale can only be formed by the combination of $(pn_-)n_+^\mu(p'n_-)n_{+\mu}$, but n_+^2 is equal to zero, and all hard loops vanish. One concludes that the Lagrangian derived above is not renormalized and complete [1]. For completeness the Yang Mill Lagrangian is derived.

3.4 The Yang-Mills Lagrangian.

Before the field redefinition (3.30) the Yang-Mills part of the Lagrangian of the effective theory is the same as in QCD with A replaced by $A_c + A_{\rm us}$. The Yang-Mills Lagrangian ca be written as:

$$\mathcal{L}_{\rm YM} = -\frac{1}{2}\,{\rm tr}\left(G_c^{\mu\nu}G^c_{\mu\nu}\right) - {\rm tr}\left(G_c^{\mu\nu}F^{\rm s}_{\mu\nu}(x)\right) - \frac{1}{2}\,{\rm tr}\left(F_{\rm s}^{\mu\nu}(x)F^{\rm s}_{\mu\nu}(x)\right) \tag{3.44}$$

with the definition

$$G_c^{\mu\nu} = [D_{\rm s}^\mu(x), A_c^\nu] - [D_{\rm s}^\nu(x), A_c^\mu] - ig\,[A_c^\mu, A_c^\nu]. \tag{3.45}$$

The first two terms of (3.44) contain products of collinear and soft fields and hence, are multipole-expanded. The third term is the soft Yang-Mills Lagrangian, which contributes a leading power term to the action.

Now, the redefinition of the field (3.30) is inserted. The Lagrangian up to λ^2 looks:

$$\mathcal{L}_{\rm YM} = -\frac{1}{2}\,{\rm tr}\left(F_c^{\mu\nu}F^c_{\mu\nu}\right) - \frac{1}{2}\,{\rm tr}\left(F_{\rm s}^{\mu\nu}(x)F^{\rm s}_{\mu\nu}(x)\right) + \mathcal{L}^{(1)}_{\rm YM} + \mathcal{L}^{(2)}_{\rm YM}, \tag{3.46}$$

where the collinear field strength tensor $F^c_{\mu\nu}$ is defined by

$$\begin{aligned} g n_{+\mu} n_{-\nu} F_c^{\mu\nu} &\equiv [n_+ D_c, in_- D], \qquad & g F_c^{\mu_\perp\nu_\perp} &\equiv [D_c^{\mu_\perp}, iD_c^{\nu_\perp}], \\ g n_{+\mu} F_c^{\mu\nu_\perp} &\equiv [n_+ D_c, iD_c^{\nu_\perp}], \qquad & g n_{-\mu} F_c^{\mu\nu_\perp} &\equiv [n_- D, iD_c^{\nu_\perp}]. \end{aligned} \tag{3.47}$$

This definition almost coincides with the standard one except that it contains $n_- D$ rather than $n_- D_c$, which is related to the presence of $A_{\rm s}$ in the collinear transformation of $n_-\hat{A}_c$ in (3.27). The first and second order terms are given by:

$$\mathcal{L}^{(1)}_{\rm YM} = {\rm tr}\left(n_+^\mu F^c_{\mu\nu_\perp} W_c\, i\Big[x_\perp^\rho n_-^\sigma F^{\rm s}_{\rho\sigma}, W_c^\dagger[iD_c^{\nu_\perp}W_c]\Big] W_c^\dagger\right) - {\rm tr}\left(n_{+\mu}F_c^{\mu\nu_\perp}W_c n_-^\rho F^{\rm s}_{\rho\nu_\perp}W_c^\dagger\right), \tag{3.48}$$

$$\begin{aligned} \mathcal{L}^{(2)}_{\rm YM} = {}& \frac{1}{2}\,{\rm tr}\left(n_+^\mu F^c_{\mu\nu_\perp} W_c\, i\Big[n_- x\, n_+^\rho n_-^\sigma F^{\rm s}_{\rho\sigma} + x_\perp^\rho x_{\perp\omega} n_-^\sigma [D_{\rm s}^\omega, F^{\rm s}_{\rho\sigma}], W_c^\dagger[iD_c^{\nu_\perp}W_c]\Big] W_c^\dagger\right) \\ & - \frac{1}{2}\,{\rm tr}\left(n_{+\mu}F_c^{\mu\nu_\perp} W_c\, i\Big[x_\perp^\rho n_-^\sigma F^{\rm s}_{\rho\nu_\perp}, W_c^\dagger in_- D W_c - in_- D_{\rm s}\Big] W_c^\dagger\right) \\ & + {\rm tr}\left(F_c^{\mu_\perp\nu_\perp} W_c\, i\Big[x_\perp^\rho F^{\rm s}_{\rho\mu_\perp}, W_c^\dagger[iD_{c\nu_\perp}W_c]\Big] W_c^\dagger\right) \\ & + \frac{1}{2}\,{\rm tr}\left(n_+^\mu n_-^\nu F^c_{\mu\nu} W_c n_+^\rho n_-^\sigma F^{\rm s}_{\rho\sigma} W_c^\dagger\right) - {\rm tr}\left(F_c^{\mu_\perp\nu_\perp}W_c F^{\rm s}_{\mu_\perp\nu_\perp}W_c^\dagger\right) \\ & - {\rm tr}\left(n_{+\mu}F_c^{\mu\nu_\perp}W_c n_-^\rho x_{\perp\sigma}[D_{\rm s}^\sigma, F^{\rm s}_{\rho\nu_\perp}]W_c^\dagger\right). \end{aligned} \tag{3.49}$$

[1] a more rigorous proof can be found in [21]

3.5 Heavy-to-Light Current to Order λ^2

In this section, the result for the representation of colour-singlet currents $\bar{\psi}\Gamma Q$ in the effective theory, where Γ is an arbitrary Dirac matrix will be discussed. These currents appear in weak decays of heavy quarks Q into light quarks. The matching to SCET is relevant in the kinematical region where large momentum is transferred from the heavy quark to the final state light quarks and gluons.

Here, the heavy to light currents are derived at tree level up to correction of order λ^2. The emission of a collinear gluon from the heavy quark puts the heavy quark far off mass-shell by an amount of order m^2. The heavy quark is described by HQET, and therefore this sort of interaction can not be explained at the Lagrangian level, but has to be reproduced by the effective current. After the collinear gluon is emitted from the heavy quark, the heavy quark stays off-shell until the heavy quark decays into a light quark. The infinite number of tree diagrams that correspond to the emission of collinear and soft gluons has to be summed into the effective current, which in position space is represented by:

$$\begin{aligned} J_{\mathrm{QCD}}(x) &= \bar{\psi}(x)\,\Gamma \sum_{n=0}^{\infty} \int dz_1 \cdots dz_n\, D_F(x-z_1)\, ig\not{A}(z_1) \cdots D_F(z_{n-1}-z_n)\, ig\not{A}_c(z_n)\, Q(z_n) \\ &= \bar{\psi}(x)\,\Gamma \sum_{n=0}^{\infty} \frac{1}{i\not{\partial}-m}\Big(-g\not{A}(x)\Big) \cdots \frac{1}{i\not{\partial}-m}\Big(-g\not{A}_c(x)\Big)\, Q(x) \\ &= e^{-im\,v\cdot x}\;\bar{\psi}\,\Gamma\left(1-\frac{1}{i\not{D}-m\,(1-\not{v})}\,g\not{A}_c\right) Q_v, \end{aligned} \tag{3.50}$$

where D_F is the heavy-quark propagator and $D^\mu = \partial_\mu - igA_\mu$ contains collinear and soft gluons, $A = A_c + A_s$. The collinear field next to the heavy quark puts the heavy quark off-shell. The field Q_v is defined in (2.3). The collinear gluon field A_c^μ can be written in a manifest gauge invariance way.

$$g\,\not{A}_c\,Q_v = [i\not{D} - m\,(1-\not{v})]\,Q_v - [i\not{D}_{\mathrm{us}} - m\,(1-\not{v})]\,Q_v = [i\not{D} - m\,(1-\not{v})]\,Q_v \tag{3.51}$$

where the equation of motion of the heavy quark has been used. Fields in terms of the effective ones are inserted after the redefinition (3.30) is performed. Expanding the currents using the method present above yield:

$$\left[\bar{\psi}(x)\,\Gamma\,Q(x)\right]_{\mathrm{QCD}} = e^{-imv\cdot x}\left\{J^{(A0)} + J^{(A1)} + J^{(A2)} + J^{(B1)} + J^{(B2)}\right\} \tag{3.52}$$

with

$$J^{(A0)} = \bar{\xi}\,\Gamma W_c h_v, \tag{3.53}$$

$$J^{(A1)} = \bar{\xi}\,\Gamma W_c\, x_{\perp\mu} D_{\mathrm{s}}^\mu h_v - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma W_c h_v, \tag{3.54}$$

$$\begin{aligned} J^{(A2)} &= \bar{\xi}\,\Gamma W_c\left(\frac{1}{2}\,n_-x\,n_+D_{\mathrm{s}}h_v + \frac{1}{2}\,x_{\mu\perp}x_{\nu\perp}D_{\mathrm{s}}^\mu D_{\mathrm{s}}^\nu h_v + \frac{i\not{D}_{\mathrm{us}}}{2m}\,h_v\right) \\ &- \bar{\xi}\,\Gamma\frac{1}{in_+D_c}\left[in_-D\,W_c - W_c\,in_-D_{\mathrm{s}}\right]h_v - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma W_c\, x_{\perp\mu} D_{\mathrm{s}}^\mu h_v, \end{aligned} \tag{3.55}$$

$$J^{(B1)} = -\bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\,[i\not{D}_{\perp c}W_c]\,h_v, \tag{3.56}$$

$$\begin{aligned} J^{(B2)} &= -\bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\,[i\not{D}_{\perp c}W_c]\,x_{\mu\perp}D_{\rm s}^{\mu}h_v - \bar{\xi}\,\Gamma\,\frac{\not{n}_-}{2m}\,[in_-D\,W_c - W_c\,in_-D_{\rm s}]\,h_v \\ &- \bar{\xi}\,\Gamma\frac{1}{in_+D_c}\left[\frac{i\not{D}_{\perp c}i\not{D}_{\perp c}}{m}W_c\right]h_v + \bar{\xi}\,i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma\frac{\not{n}_-}{2m}\,[i\not{D}_{\perp c}W_c]\,h_v, \end{aligned} \tag{3.57}$$

Remember that derivatives operate on soft fields before $x = x_-$ is set, and that derivatives with square brackets act only within the brackets.

In the next Chapter heavy to light transitions in the HQET framework are going to be studied.

Chapter 4

Heavy to Light Currents

Contents

Heavy to light currents are of great importance since they appear in many composite operator coming from the weak interaction and contain in their coefficients some of the parameters of the CKM matrix elements. When taking matrix elements of them, one has to deal with the confinement problem of the QCD interactions.

Here, heavy to light transitions are going to be considered in the limit in which the mass of the heavy quark is large in comparison with the soft momenta carried by the light components. In this limit, one can use HQET in order to deal with the QCD interactions.

The full QCD operators are expanded in a $1/m$ series, and their coefficients are HQET operators with the corresponding dimensionality and quantum numbers. The effective operators describe the low energy physics and are multiplied by Wilson coefficients which account for the short distance interactions.

Matrix elements in the full an in the effective theory are parametrized in terms of non-perturbative universal form factors, non computable from first principles. The simplest one, matrix elements between the vacuum and B or B^* meson states, define meson decays constants which appear in many processes, i.e., in $B \to \nu_l l$. This decay, on case of better luminosity, will allow the extraction of V_{ub}.

In the effective theory symmetries appear and at leading order only one function parametrizes the non-perturbative physics. Therefore, ratios of meson decays constants, such as f_{B^*}/f_B, are given by ratios of the corresponding Wilson coefficients, which are computable perturbatively.

In order to compete with the experimental results, one needs to include symmetry-breaking subleading terms in the $1/m$ expansion, and also radiative corrections. Previous analyses of heavy to light currents have been performed on this direction [111–113]. Here in this chapter a full next-to-leading analyses will be performed for an arbitrary Dirac structure.

The result of the $1/m$ expansion of the dimension 3 full QCD currents for an arbitrary Dirac structure will be collected in Sect. 4.1, including reparametrization constrains and the one loop corrections for the Wilson coefficient of the subleading 4 dimension HQET operator. Furthermore, it will be shown that only four different currents are needed in order to describe a generic heavy-to light transition, two spin-0 currents and two spin-1 currents. In Sect. 4.2, the renormalization of the dimension 4 operator will be presented. In Sect. 4.3 and Sect. 4.4, the Spin 0 and Spin 1 currents will be studied in detail. Ratios of matrix elements such as f_B and f_{B^*} are given at leading order by ratios of the corresponding matching coefficients. The full next-to-leading analysis of the meson decay constants including $1/m$ correction expressed in terms of invariant renormalization constants will be given in Sect. 4.3 and Sect. 4.4. In these sections, it can be seen that matrix elements of the 4-dimension operator receives non-trivial radiative corrections. This will establish the basis to study the asymptotic behaviour of the leading order matching coefficients which will be presented in the next Chapter.

4.1 Heavy to Light Currents

The dimension three heavy to light current in terms of the bare quarks fields is written as:

$$j_0 = \bar{q}_0 \Gamma Q_0 \tag{4.1}$$

where Γ is an antisymmetrized product of n Dirac γ matrices

$$\Gamma = \gamma_\perp^{[\alpha_1} \cdots \gamma_\perp^{\alpha_n]} \quad \text{or} \quad \gamma_\perp^{[\alpha_1} \cdots \gamma_\perp^{\alpha_{n-1}]} \not{v}\,, \tag{4.2}$$

which commutes or anticommutes with $\not{v}$:

$$\not{v}\Gamma = \sigma \Gamma \not{v}\,, \quad \sigma = \pm 1\,,$$

and $\gamma_\perp^\alpha = \gamma^\alpha - \not{v} v^\alpha$. The bare currents can be written in terms of the renormalized currents by $j_0 = \bar{q}_0 \Gamma Q_0 Z_j'(\alpha_s'(\mu)) j_\Gamma(\mu)$. Here $\alpha_s'(\mu)$ is the QCD coupling with $n_f = n_l + 1$ flavours. $Z_j'(\alpha_s'(\mu))$ is the renormalization constant and $j(\mu)$ are renormalized currents which receives contributions from a large range of energy scales, starting at $\mu = m_b$ for a b quark at $\mu \equiv m_b$, down to $\mu \equiv \Lambda_{QCD}$. j_0 does not depend on the parameter μ, and hence follows the renormalization group equation

$$\left(\frac{d}{d \ln \mu} + \gamma_\Gamma' \left(\alpha_s'(\mu) \right) \right) j_\Gamma(\mu) = 0, \tag{4.3}$$

where

$$\gamma_\Gamma' \left(\alpha_s'(\mu) \right) = \frac{d \log Z_\Gamma}{d \log \mu} \tag{4.4}$$

is the anomalous dimension and is known up to three loops [114]. Up to two loops [50]:

$$\begin{aligned} \gamma_\Gamma' = & -2(n-1)(n-3) C_F \frac{\alpha_s}{4\pi} \\ & \times \left\{ 1 + \left[\tfrac{1}{2}(5(n-2)^2 - 19) C_F - \tfrac{1}{3}(3(n-2)^2 - 19) C_A \right] \frac{\alpha_s}{4\pi} \right\} \\ & - \tfrac{1}{3}(n-1)(n-15) C_F \beta_0' \left(\frac{\alpha_s}{4\pi} \right)^2 + \cdots \end{aligned} \tag{4.5}$$

where n is the number of the gamma matrices. Note that γ'_Γ vanishes for the vectorial case, $n = 1$.

In HQET, at leading order, there is only a single current

$$\tilde{\jmath}_0 = \bar{q}_0 \Gamma h_{v0} = \tilde{Z}_j(\alpha_s(\mu))\tilde{\jmath}(\mu) \tag{4.6}$$

where the heavy quark has been replaced by the effective heavy quark h_{v0} which is renormalized in the heavy quark effective theory and carries momenta of order Λ_{QCD}. $\alpha_s(\mu)$ is the QCD coupling with n_l flavours and the dependence with μ is given by the renormalization group equation:

$$\left(\frac{d}{d\ln\mu} + \tilde{\gamma}\left(\alpha_s(\mu)\right)\right)\tilde{\jmath}(\mu) = 0, \qquad \tilde{\gamma} = \frac{d\log\tilde{Z}_j}{d\log\mu} \tag{4.7}$$

the anomalous dimension $\tilde{\gamma}$ does not depend on Γ [8,31,115–117].

$$\begin{aligned}\tilde{\gamma} &= -3C_F\frac{\alpha_s}{4\pi} \\ &+ C_F\left[\left(-\frac{8}{3}\pi^2 + \frac{5}{2}\right)C_F + \left(\frac{2}{3}\pi^2 - \frac{49}{6}\right)C_A + \frac{10}{3}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots\end{aligned} \tag{4.8}$$

QCD currents are expanded in a $1/m$ expansion; the coefficient of this expansion are HQET operators with the appropriate quantum numbers and dimensionality. At tree level this is given by the expansion of the full heavy quark field in terms of the effective one:

$$j = \bar{q}\Gamma h_v + \frac{1}{2m_b}\bar{q}\Gamma \not{D} h_v \tag{4.9}$$

Beyond tree level the low energy effective field theory, HQET, does not describe the short distance interactions properly and these are corrected by Wilson coefficients:

$$j(\mu') = C_\Gamma(\mu',\mu)\tilde{\jmath}(\mu) + \mathcal{O}(1/m)\,. \tag{4.10}$$

where $C_\Gamma(\mu',\mu)$ are the short distance Wilson coefficients. In doing the matching the logarithmic m dependence of the QCD currents has been isolated in the Wilson coefficients. It is more natural to perform the matching at $\mu' = \mu = m$, where the matching coefficients do not contain large logarithms. For an arbitrary normalization scale, applying the renormalization group equations (4.3) and (4.7)

$$\begin{aligned}C_\Gamma(\mu',\mu) &= \hat{C}_\Gamma\left(\frac{\alpha'_s(\mu')}{\alpha'_s(\mu_0)}\right)^{\frac{\gamma'_{\Gamma 0}}{2\beta'_0}}\left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{\tilde{\gamma}_0}{2\beta_0}} K'_{\gamma'_\Gamma}(\alpha'_s(\mu'))K_{-\tilde{\gamma}}(\alpha_s(\mu))\,, \\ \hat{C}_\Gamma &= C_\Gamma(m,m)K'_{-\gamma'_\Gamma}(\alpha'_s(m))K_{\tilde{\gamma}}(\alpha_s(m))\,,\end{aligned} \tag{4.11}$$

which relates currents at arbitrary normalization scale. Here K' involves the n_f-flavour β-function β'. The full one-loop corrections to $C_\gamma(m,m)$ were obtained in [10], and two loops ones in [50, 118]. $\hat{C}_\Gamma$ are renormalization group invariant constants and are given by perturbation series in $\alpha(\mu_0)$:

$$\hat{C}_\Gamma = 1 + \sum_{L=1}^{\infty} c_L^\Gamma\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L\,, \tag{4.12}$$

$$
\begin{aligned}
c_1^\Gamma = C_F \Big\{ & \tfrac{3}{2}(n-2)^2 - \eta(n-2) - \tfrac{13}{4} \\
& + \left[\left(-\tfrac{4}{3}\pi^2 + \tfrac{23}{4}\right) C_F + \left(\tfrac{1}{3}\pi^2 + 8\right) C_A\right] \frac{1}{\beta_0} - \tfrac{3}{2}\left(11C_F + 7C_A\right) \frac{C_A}{\beta_0^2} \\
& + \left[\tfrac{5}{2}\left((n-2)^2 - 5\right) C_F - \tfrac{1}{3}\left(3(n-2)^2 - 4\right) C_A\right] (n-1)(n-3) \frac{1}{\beta_0'} \\
& + \left(11C_F + 7C_A\right)(n-1)(n-3) \frac{C_A}{\beta_0'^2} \Big\} ,
\end{aligned}
$$

where $\eta = -\sigma(-1)^n$ and $\mu = me^{-5/6}$. It is possible to obtain c_2^Γ from the known two-loop results for $C_\Gamma(m,m)$ since $\tilde{\gamma}_2$ has been calculated recently [119]. In the next Chapter, the behaviour of the coefficients c_L^Γ at $L \gg 1$ will be investigated using a renormalon analysis.

There are various prescriptions for handling γ_5 in dimensional regularization. Multiplying Γ by the anticommuting $\gamma_5^{\rm AC}$ does not change C_Γ. This is not true for the 't Hooft–Veltman $\gamma_5^{\rm HV}$. QCD currents with $\gamma_5^{\rm AC}$ and $\gamma_5^{\rm HV}$ are related by finite renormalization factors [120–123]:

$$
\left(\bar{q}\Gamma\gamma_5^{\rm AC}Q\right)_{\mu'} = K'_{\gamma'_{\Gamma\gamma_5^{\rm AC}} - \gamma'_{\Gamma\gamma_5^{\rm HV}}}(\alpha'_s(\mu')) \left(\bar{q}\Gamma\gamma_5^{\rm HV}Q\right)_{\mu'} , \tag{4.13}
$$

where the anomalous dimensions $\gamma'_{\Gamma\gamma_5^{\rm AC}} = \gamma'_\Gamma$ and $\gamma'_{\Gamma\gamma_5^{\rm HV}}$ differ starting from two loops. In HQET, both currents have the same anomalous dimension $\tilde{\gamma}$, and hence the similar renormalization factor is unity. Therefore, $C_{\Gamma\gamma_5^{\rm HV}}(\mu',\mu)$ differs from $C_\Gamma(\mu',\mu)$ only by $K'_{\gamma'_\Gamma}(\alpha'_s(\mu'))$ in (4.11), and $\hat{C}_{\Gamma\gamma_5^{\rm HV}} = \hat{C}_\Gamma$. For $\sigma^{\alpha\beta}$, multiplication by $\gamma_5^{\rm HV}$ is just a Lorentz rotation, and does not change the anomalous dimension. Therefore, $\left(\bar{q}\sigma^{\alpha\beta}\gamma_5^{\rm AC}Q\right)_{\mu'} = \left(\bar{q}\sigma^{\alpha\beta}\gamma_5^{\rm HV}Q\right)_{\mu'}$, and $C_{\sigma_\perp}(\mu',\mu) = C_{\gamma_\perp\not{v}}(\mu',\mu)$ [50], where $\sigma_\perp^{\alpha\beta} = \frac{i}{2}[\gamma_\perp^\alpha, \gamma_\perp^\beta]$.

There are 8 different Dirac matrices Γ (4.2) in 4-dimensional space. For our investigation of $\hat{C}_\Gamma$, one can restrict the basis to

$$
\Gamma = 1 , \quad \not{v} , \quad \gamma_\perp^\alpha , \quad \gamma_\perp^\alpha \not{v} , \tag{4.14}
$$

because the other 4 matrices can be obtained from (4.14) by multiplying by $\gamma_5^{\rm HV}$. Non-vanishing matrix elements between B or B^* and the vacuum are:

$$
\begin{aligned}
\langle 0|(\bar{q}\gamma_5^{AC}Q)_\mu|B\rangle &= -im_B f_B^P(\mu), \\
\langle 0|\bar{q}\gamma^\alpha\gamma_5^{AC}Q|B\rangle &= ip^\alpha f_B, \\
\langle 0|\bar{q}\gamma^\alpha Q|B^*\rangle &= im_{B^*} f_{B^*} e^\alpha, \\
\langle 0|(\bar{q}\sigma_{\alpha\beta}Q)_\mu|B^*\rangle &= f_{B^*}^T(\mu)(p^\alpha e^\beta - p^\beta e^\alpha),
\end{aligned} \tag{4.15}
$$

where e is the B^* polarization vector. Ommitting γ_5^{AC} factors, the first matrix element is studied with $\Gamma = 1$, the second with $\Gamma = \not{v}$ after contracting with v^α, corresponding to Spin 0 current transitions. The last ones are studied with $\Gamma = \gamma_{\perp\mu}$ and $\Gamma = \gamma_\perp^\beta \not{v}$ respectively, after contracting with e_μ, since the longitudinal part vanish, and with $v^\alpha e^\beta$. These correspond to Spin 1 current transitions. In turn, the

HQET matrix element due to heavy quark and spin symmetries can be described in a compact way:

$$\langle 0|(\bar{q}\Gamma h_v)_\mu|M\rangle = \frac{F(\mu)}{2}\mathrm{Tr}[\Gamma\mathcal{M}] \tag{4.16}$$

M is the effective B meson state, $F(\mu)$ is the effective B meson decay constant and due to flavour spin symmetry only one is required. $\mathcal{M}$ is the spin wave function of the meson M, for the ground state:

$$\mathcal{M} = \sqrt{m_M}\frac{1+\not{v}}{2}\begin{cases} -i\gamma_5; & J^P = 0^-, \\ \not{\epsilon}; & J^P = 1^-, \end{cases} \tag{4.17}$$

Therefore ratios as f_B/f_{B^*} are given at leading order by ratios of computable Wilson coefficients.

$$\frac{f_{B^*}}{f_B} = \frac{\hat{C}_{\gamma_\perp}}{\hat{C}_{\not{v}}}. \tag{4.18}$$

Corrections to the leading order current appears as higher order dimension operators [124]:

$$j(\mu') = C_\Gamma(\mu',\mu)\tilde{\jmath}(\mu) + \frac{1}{2m}\sum_i B_i^\Gamma(\mu',\mu)O_i(\mu) + \mathcal{O}(1/m^2)\,. \tag{4.19}$$

O_i are 4 dimension local operators with the appropriate quantum numbers, when taking matrix elements to order $1/m$, one can either take the leading order HQET Lagrangian with the subleading operator or the leading order current with the subleading terms of the HQET Lagrangian. Here, the latter terms are included in the expansion by adding the bilocal operators [92]:

$$C_\Gamma \int dx\, i\,T\left\{\tilde{\jmath}(0), O_k(x) + C_m O_m(x)\right\}\,. \tag{4.20}$$

where O_k is the kinetic operators which is not renormalized due to RPI, and O_m is the chromomagnetic operator. The Wilson coefficients of these bilocal operators will be given in terms of products of known Wilson coefficients. The full one-loop corrections to B_i for vector currents (and axial currents with anticommuting γ_5) were given in [125,126]. Some general properties of the matching coefficients B_i and the anomalous dimension matrix of O_i following from reparametrization invariance and equations of motion were established in [126], and the two-loop anomalous dimension matrix was calculated in [112,113].

The dimension 4 local operators are of the form $\bar{q}\overleftarrow{D}h_v$ and $\bar{q}Dh_v$. The terms in the sum with the derivative acting on the heavy-quark field can be obtained from reparametrization invariance [126]. Let $\Gamma = \gamma^{[\alpha_1}\cdots\gamma^{\alpha_n]}$. It can be decomposed into the parts commuting and anticommuting with $\not{v}$:

$$\Gamma = \Gamma_+ + \Gamma_-\,, \quad \Gamma_\pm = \frac{1}{2}\left(\Gamma \pm \not{v}\Gamma\not{v}\right)\,.$$

The matrix element of the renormalized QCD current $\bar{q}\Gamma Q$ from the heavy-quark state with momentum mv to the light-quark state with momentum 0 is

$$\tfrac{1}{2}\left(C_{\Gamma_+} + C_{\Gamma_-}\right)\bar{u}_q\Gamma u(mv) + \tfrac{1}{2}\left(C_{\Gamma_+} - C_{\Gamma_-}\right)\bar{u}_q\not{v}\Gamma u(mv)\,. \tag{4.21}$$

In this equation, substituting $v \to v + k/m$,

$$\tfrac{1}{2}\left(C_{\Gamma_+} + C_{\Gamma_-}\right)\bar{u}_q \Gamma u(mv+k) \\ + \tfrac{1}{2}\left(C_{\Gamma_+} - C_{\Gamma_-}\right)\bar{u}_q \left(\not{v} + \frac{\not{k}}{m}\right)\Gamma u(mv+k)\,.$$

and using $u(mv+k) = (1 + \not{k}/(2m))\, u_v(k)$, one obtains the leading term (4.21) plus

$$\frac{1}{4m}\left[\left(C_{\Gamma_+} + C_{\Gamma_-}\right)\bar{u}_q \Gamma \not{k} u_v + \left(C_{\Gamma_+} - C_{\Gamma_-}\right)\bar{u}_q \left(\not{v}\Gamma\not{k} + 2\not{k}\Gamma\right) u_v\right]\,.$$

Therefore, the $\bar{q}Dh_v$ terms in the sum in (4.19) are

$$\tfrac{1}{2}\left(C_{\Gamma_+} + C_{\Gamma_-}\right)\bar{q}\Gamma i\not{D} h_v + \tfrac{1}{2}\left(C_{\Gamma_+} - C_{\Gamma_-}\right)\bar{q}\left(\not{v}\Gamma i\not{D} + 2i\not{D}\Gamma\right)h_v\,. \tag{4.22}$$

The coefficients of operators with the derivative acting on the light-quark field are not determined by general considerations. These coefficients appear first at the one-loop level. To obtain them, the matrix element of the QCD current from the heavy quark with momentum mv to the light quark with momentum p (with $p^2 = 0$), expanded in p/m to the linear term, and equate it to the corresponding HQET matrix element is calculated. In HQET, loop corrections contain no scale, and hence vanish (except, possibly, massive-quark loops, which first appear at the two-loop level). The QCD matrix element is proportional to $\bar{u}(p)\Gamma(p,mv)u(mv)$, where $\Gamma(p,mv)$ is the bare proper vertex function. At one loop, it is given by Fig. 4.1. the

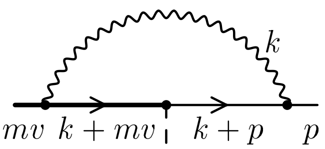

Figure 4.1: One-loop matching

term linear in p has the structure:

$$\bar{u}(p)\left[\sum_i x_i L_i \Gamma R_i\right] u(mv) \tag{4.23}$$

with $L_i \times R_i = p\cdot v\, 1 \times 1,\ p\cdot v\, \not{v}\gamma_\mu \times \gamma^\mu,\ p\cdot v\, \gamma_\mu\gamma_\nu \times \gamma^\nu\gamma^\mu,\ 1 \times \not{p},\ \not{v}\gamma_\mu \times \gamma^\mu\not{p}$. The coefficients x_i can be obtained by solving a linear system, from the double traces of Dirac matrices to the left from Γ with $\bar{L}_j$ and those to the right from Γ with $\bar{R}_j$, with $\bar{L}_j \times \bar{R}_j = \not{p}\not{v} \times (1+\not{v}),\ \gamma_\rho\not{v} \times (1+\not{v})\gamma^\rho,\ \gamma_\rho\gamma_\sigma\not{p}\not{v} \times (1+\not{v})\gamma^\sigma\gamma^\rho$, $\not{p}\not{v}\times(1+\not{v})\not{p},\ \gamma_\rho\not{v}\times(1+\not{v})\not{p}\gamma^\rho$. Now one can take these double traces of the *integrand* of Fig. 4.1, and express x_i via scalar integrals. Their numerators involve $(k\cdot p)^n$; putting $k = (k\cdot v)v + k_\perp$ and averaging over $k_\perp$ directions in the $(d-1)$-dimensional subspace, one can express them via the factors in the denominator.

Now assuming:

$$\not{v}\Gamma = \sigma\Gamma\not{v}\,,\quad \sigma = \pm 1\,,\quad \gamma_\mu\Gamma\gamma^\mu = 2\sigma h\Gamma\,. \tag{4.24}$$

For the Γ matrices (4.2),

$$h = \eta \left(n - \frac{d}{2} \right) , \quad \eta = -\sigma(-1)^n . \tag{4.25}$$

Then (4.23) becomes

$$\begin{aligned} &\left[x_1 + (x_2 + 2x_5) \cdot 2h + x_3 (2h)^2\right] \; p \cdot v \, \bar{u}(p) \Gamma u(mv) \\ &+ [x_4 - x_5 \cdot 2h] \, \bar{u}(p) \Gamma \not{p} u(mv) . \end{aligned} \tag{4.26}$$

Performing the simple calculation, the result is:

$$\begin{aligned} \bar{u}(p)\Gamma(p, mv) u(mv) &= \left[1 + C_F \frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) c_\Gamma \right] \bar{u}(p) \Gamma u_v(0) \\ + \frac{1}{2m} C_F \frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}} \Gamma(\varepsilon) &\left[b_1^\Gamma \, p \cdot v \, \bar{u}(p) \Gamma u_v(0) + b_2^\Gamma \bar{u}(p) \Gamma \not{p} u_v(0) \right] , \end{aligned} \tag{4.27}$$

where

$$\begin{aligned} c_\Gamma &= -\frac{(1-h)(d-2+2h)}{(d-2)(d-3)} , \\ b_1^\Gamma &= -2 \frac{(d-2)(d-8) - (d-5)(d-4+2h)h}{(d-2)(d-3)(d-5)} , \\ b_2^\Gamma &= 2 \frac{d-2-h}{(d-2)(d-3)} . \end{aligned}$$

The zeroth-order coefficient c_Γ has been found in [10, 50]; the first-order coefficients for components of the vector current in [125, 126].

For $\Gamma = 1$, $\not{v}$, the square bracket in (4.27) becomes $b_\Gamma \, p \cdot v \, \bar{u}(p) \Gamma u_v(0)$, with $b_1 = b_1^1$, $b_{\not{v}} = b_1^{\not{v}} + 2 b_2^{\not{v}}$. These results have been checked by taking the trace of the whole integrand of Fig. 4.1 with $(1+\not{v})\not{p}$ and calculating the integrals. For $\Gamma = \gamma_\perp^\alpha$, $\gamma_\perp^\alpha \not{v}$, the square bracket in (4.27) becomes $b_{\Gamma,1} \, p \cdot v \, \bar{u}(p) \gamma_\perp^\alpha u_v(0) + b_{\Gamma,2} p_\perp^\alpha \bar{u}(p) u_v(0)$, with $b_{\gamma_\perp,1} = b_1^{\gamma_\perp}$, $b_{\gamma_\perp,2} = 2 b_2^{\gamma_\perp}$, $b_{\gamma_\perp \not{v},1} = b_1^{\gamma_\perp \not{v}} + 2 b_2^{\gamma_\perp}$, $b_{\gamma_\perp \not{v},2} = -2 b_2^{\gamma_\perp \not{v}}$. These results have been checked by taking traces of the integrand with $(1+\not{v}) p_\alpha \not{p}$ and $(1+\not{v}) \gamma_\alpha \not{p}$. An additional strong check is provided by the Ward identity: contracting the vertex function $\Gamma^\alpha(p, mv)$ (for $\Gamma = \gamma^\alpha$) with the momentum transfer $(mv - p)_\alpha$, one obtains

$$\Gamma^\alpha(p, mv)(mv - p)_\alpha = m \Gamma(p, mv) + \Sigma(mv) ,$$

where $\Gamma(p, mv)$ is the scalar vertex (for $\Gamma = 1$), and Σ is the heavy-quark self-energy. At the first order in p, this leads to

$$b_1 - b_{\not{v}} = 2 \left(c_{\gamma_\perp} - c_{\not{v}} \right) . \tag{4.28}$$

The results (4.27) satisfy this requirement.

In the next Section, the renormalization of the four dimension operators will be discussed.

4.2 Renormalization of dimension-4 operators

In order to study the renormalization of the dimension four operators it is convenient to choose a set of operators, which close under renormalization. That is, a set of bare operators O_0^i are renormalized to a combination of the same set of renormalize operators $O(\mu)_i$:

$$O_0^i = Z(\mu)_{ij} O(\mu)_j \tag{4.29}$$

where $Z(\mu)_{ij}$ is the matrix of the renormalization constants. The renormalized operators follows the renormalization group equations,

$$\frac{dO(\mu)_i}{d\log\mu} + \gamma_{ij}(\alpha_s(\mu))O(\mu)_j = 0 \tag{4.30}$$

where the anomalous dimension matrix γ is determined by the matrix of renormalized constants Z

$$\gamma = \frac{\log Z(\mu)}{\log\mu} = -2\frac{\log Z(\mu)}{\log\alpha_s}(\epsilon + \beta(\alpha_s(\mu))) \tag{4.31}$$

where $(d\log\alpha/d\log\mu) = -2(\epsilon + \beta(\mu))$ has been used in the last step. Z can be written as:

$$\log Z = 1 + \frac{Z_1}{\epsilon} + \frac{Z_2}{\epsilon^2} + \ldots \tag{4.32}$$

Since the anomalous dimension should be finite when $\epsilon \to 0$, one can identify:

$$\gamma = -2\frac{dZ_1}{d\log\alpha_S}, \qquad \frac{dZ_2}{d\log\alpha_S} = (Z_1 - \beta(\alpha_S))\frac{dZ_1}{d\log\alpha_S} \tag{4.33}$$

Hence, the matrix of renormalization constant can be expressed in terms of matrix of anomalous dimension γ. Up to two loop accuracy:

$$\begin{aligned} Z &= 1 + \frac{1}{2}\gamma_0\frac{\alpha_S}{4\pi\epsilon} + \frac{1}{8}\left[(\gamma_0 + 2\beta_0)\gamma_0 - 2\gamma_1\epsilon\right]\left(\frac{\alpha_S}{4\pi\epsilon}\right)^2 \\ Z^{-1} &= 1 + \frac{1}{2}\gamma_0\frac{\alpha_S}{4\pi\epsilon} + \frac{1}{8}\left[(\gamma_0 + 2\beta_0)\gamma_0 - 2\gamma_1\epsilon\right]\left(\frac{\alpha_S}{4\pi\epsilon}\right)^2 \end{aligned} \tag{4.34}$$

First, the local operators are studied. As discussed in [112], three operators

$$\begin{aligned} &O_1^\Gamma = \bar{q} i D^\alpha \Gamma h_v\,, \quad O_2^\Gamma = \bar{q}\left(-i\overleftarrow{D}_\perp^\alpha\right)\Gamma h_v\,, \\ &O_3^\Gamma = \bar{q}\left(-iv\cdot\overleftarrow{D}\right)\gamma_\perp^\alpha \not{v}\Gamma h_v = -iv\cdot\partial\left(\bar{q}\gamma_\perp^\alpha\not{v}\Gamma h_v\right) \end{aligned} \tag{4.35}$$

close under renormalization for any Γ. Two operators are full derivatives of the leading order current which is renormalized by $\tilde{Z}$. Therefore, they renormalize by: $O_{30}^\Gamma = \tilde{Z} O_3^\Gamma$, $O_{01}^\Gamma - O_{20}^\Gamma = \tilde{Z}(O_1 - O_2)$. O_1 mixes with O_2 and O_3. The non-mixing part renormalize with $\tilde{Z}$. Reparametrization invariance links O_1^Γ with the leading order current, whose ultraviolet behaviour is governed by $\tilde{\gamma}$. Therefore,

$O_0^\Gamma = Z(\alpha_s(\mu))O^\Gamma(\mu)$, where

$$Z = \tilde{Z} + \begin{pmatrix} 0 & Z_a & Z_b \\ 0 & Z_a & Z_b \\ 0 & 0 & 0 \end{pmatrix}, \quad Z^{-1} = \tilde{Z}^{-1} + \begin{pmatrix} 0 & \bar{Z}_a & \bar{Z}_b \\ 0 & \bar{Z}_a & \bar{Z}_b \\ 0 & 0 & 0 \end{pmatrix},$$
$$\gamma = \tilde{\gamma} + \begin{pmatrix} 0 & \gamma_a & \gamma_b \\ 0 & \gamma_a & \gamma_b \\ 0 & 0 & 0 \end{pmatrix}. \tag{4.36}$$

For $\Gamma = \gamma_\alpha \Gamma'$, then

$$O_{10}^\Gamma = O'_{10}, \quad O_{20}^\Gamma = O'_{20}, \quad O_{30}^\Gamma = (3-2\varepsilon)O'_{20}, \tag{4.37}$$

where

$$O'_1 = i\partial_\alpha \left(\bar{q}\gamma^\alpha \Gamma' h_v\right), \quad O'_2 = iv\cdot\partial\left(\bar{q}\not{v}\Gamma' h_v\right). \tag{4.38}$$

with $O'_0 = \tilde{Z}(\alpha_s(\mu))O'(\mu)$, since they are full derivatives of the leading order current. In terms of the renormalize finite operators:

$$O_1^\Gamma(\mu) = O'_1(\mu) + \tilde{Z}\left(\bar{Z}_a + (3-2\varepsilon)\bar{Z}_b\right)O'_2(\mu),$$
$$O_2^\Gamma(\mu) = \left[1 + \tilde{Z}\left(\bar{Z}_a + (3-2\varepsilon)\bar{Z}_b\right)\right]O'_2(\mu),$$
$$O_3^\Gamma(\mu) = 3O'_2(\mu). \tag{4.39}$$

Therefore, $\tilde{Z}\left(\bar{Z}_a + (3-2\varepsilon)\bar{Z}_b\right)$ must be finite at $\varepsilon \to 0$. This allows one to reconstruct γ_b from γ_a:

$$\gamma_b = -\frac{1}{3}\left(\gamma_a + \Delta\gamma_a\right), \quad \Delta\gamma_a = \frac{1}{3}\gamma_{a0}\left(\gamma_{a0} - 2\beta_0\right)\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3). \tag{4.40}$$

The anomalous dimensions $\gamma_{a,b}$ have been calculated in [112] to two-loop accuracy (they are called $\gamma_{2,4}$ in [112]). The result there, satisfies relation (4.40):

$$\gamma_a = 3C_F\frac{\alpha_s}{4\pi} + C_F\left[\left(\frac{4}{3}\pi^2 - 5\right)C_F\right.$$
$$\left. + \left(-\frac{1}{3}\pi^2 + \frac{41}{3}\right)C_A - \frac{10}{3}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \tag{4.41}$$

The finite parts, at the next-to-leading order, are

$$O_1^\Gamma(\mu) = O'_1(\mu) + \frac{1}{3}\gamma_{a0}\frac{\alpha_s(\mu)}{4\pi}O'_2(\mu),$$
$$O_2^\Gamma(\mu) = \left[1 + \frac{1}{3}\gamma_{a0}\frac{\alpha_s(\mu)}{4\pi}\right]O'_2(\mu),$$
$$O_3^\Gamma(\mu) = 3O'_2(\mu). \tag{4.42}$$

Note that $O_{10}^\Gamma = O'_{10}$, but $O_1^\Gamma(\mu) \neq O'_1(\mu)$: additional counterterms in (4.39) yield a finite contribution, because of the $\mathcal{O}(\varepsilon)$ term in (4.37).

Now the bilocal terms with the kinetic insertion are studied. As discussed in [113], two operators

$$O_1^k = \bar{q}\left(-iv\cdot\overleftarrow{D}\right)\Gamma h_v\,,\quad O_2^k = i\int dx\, T\left\{\bar{q}\Gamma h_v, O_k(x)\right\}$$

close under renormalization for any Γ, and

$$Z = \tilde{Z} + \begin{pmatrix} 0 & 0 \\ Z^k & 0 \end{pmatrix}\,,\quad Z^{-1} = \tilde{Z}^{-1} + \begin{pmatrix} 0 & 0 \\ \bar{Z}^k & 0 \end{pmatrix}\,,\quad \gamma = \tilde{\gamma} + \begin{pmatrix} 0 & 0 \\ \gamma^k & 0 \end{pmatrix}\,.$$

with γ^k up to two loops [113]:

$$\gamma^k = -8C_F\frac{\alpha_s}{4\pi} + C_F\left[\left(-\frac{32}{9}\pi^2 - \frac{32}{3}\right)C_F \right. \\ \left. + \left(\frac{16}{3}\pi^2 - \frac{608}{9}\right)C_A + \frac{160}{9}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \tag{4.43}$$

Finally, the bilocal operator with the chromomagnetic operator will be explained. The closed set of operators under renormalization for any Γ

$$\begin{aligned} O_1^m &= -\tfrac{1}{4}\bar{q}\left(-iv\cdot\overleftarrow{D}\right)\sigma_{\mu\nu}\Gamma(1+\not{v})\sigma^{\mu\nu}h_v\,, \\ O_2^m &= -\tfrac{1}{4}\bar{q}\left(-i\overleftarrow{D}_\nu\right)i\gamma_\mu\not{v}\Gamma(1+\not{v})\sigma^{\mu\nu}h_v\,, \\ O_3^m &= i\int dx\, T\left\{\bar{q}\Gamma h_v, O_m(x)\right\} \end{aligned} \tag{4.44}$$

Note that the indices μ, ν live in the subspace orthogonal to v, due to $\not{v}h_v = h_v$. These operators have [113]

$$Z = \begin{pmatrix} \tilde{Z} & 0 & 0 \\ Z_b & \tilde{Z}+Z_a & 0 \\ Z_b^m & Z_a^m & \tilde{Z}Z_m \end{pmatrix}\,,\quad Z^{-1} = \begin{pmatrix} \tilde{Z}^{-1} & 0 & 0 \\ \bar{Z}_b & \tilde{Z}^{-1}+\bar{Z}_a & 0 \\ \bar{Z}_b^m & \bar{Z}_a^m & \tilde{Z}^{-1}Z_m^{-1} \end{pmatrix}\,,$$

$$\gamma = \tilde{\gamma} + \begin{pmatrix} 0 & 0 & 0 \\ \gamma_b & \gamma_a & 0 \\ \gamma_b^m & \gamma_a^m & \gamma_m \end{pmatrix}\,. \tag{4.45}$$

The first two operators in (4.44) are (4.35) $O_3^{\Gamma'}$ and $O_2^{\Gamma'}$ with $\Gamma'_\alpha = -\frac{1}{4}i\gamma^\mu\not{v}\Gamma(1+\not{v})\sigma_{\mu\alpha}$; therefore, the upper left 2×2 blocks in (4.45) follow from (4.36). O_3^m needs additional counterterms from O_1^m and 0_2^m. The anomalous dimensions up to loops were calculated in [113].

$$\begin{aligned} \gamma_a^m &= -2C_F\frac{\alpha_s}{4\pi} + \\ &+ C_F\left[\left(-\frac{8}{9}\pi^2 + \frac{22}{3}\right)C_F + \left(\frac{2}{9}\pi^2 - \frac{110}{9}\right)C_A + \frac{4}{9}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 \\ \gamma_b^m &= -2C_F\frac{\alpha_s}{4\pi} + \\ &+ C_F\left[\left(-\frac{40}{9}\pi^2 - \frac{16}{3}\right)C_F + \left(\frac{10}{9}\pi^2 - \frac{50}{9}\right)C_A + \frac{4}{9}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 \end{aligned} \tag{4.46}$$

Now the heavy-to light Spin 0 currents transitions will be presented.

4.3 Spin-0 currents

In the case of the currents with $\Gamma = 1$, $\not{v}$, the leading term in the expansion (4.19) contains $\tilde{\jmath} = \bar{q}h_v$, and the $1/m$ corrections are given by four operators

$$\begin{aligned} O_1 &= \bar{q}i\not{D}h_v = i\partial_\alpha\left(\bar{q}\gamma^\alpha h_v\right) , \\ O_2 &= \bar{q}\left(-iv\cdot\overleftarrow{D}\right)h_v = -iv\cdot\partial\left(\bar{q}h_v\right) , \\ O_3 &= i\int dx\, T\left\{\tilde{\jmath}(0), O_k(x)\right\} , \\ O_4 &= i\int dx\, T\left\{\tilde{\jmath}(0), O_m(x)\right\} . \end{aligned} \tag{4.47}$$

The operators $O_{1,2}$ are renormalized multiplicatively with $\tilde{Z}$. For $\Gamma = 1$ $O^k_{20} = O_{30}$ and $O^m_{30} = O_{40}$ and hence need counterterms proportional to O_2. Therefore, the renormalization constant and the anomalous dimension matrix of the dimension-4 operators (4.47) have the structure:

$$Z = \tilde{Z} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & Z^k & 0 & 0 \\ 0 & Z^m & 0 & Z_m \end{pmatrix} , \quad \tilde{\gamma} + \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & \gamma^k & 0 & 0 \\ 0 & \gamma^m & 0 & \gamma_m \end{pmatrix} . \tag{4.48}$$

where γ^k is given in (4.43) and γ^m can be obtained from the known $\gamma^m_{a,b}$. In the case $\Gamma = 1$, the operators (4.44) are related to (4.47) by:

$$\begin{aligned} &O^m_{10} = -(1-\varepsilon)(3-2\varepsilon)O_{20} , \quad O^m_{20} = -(1-\varepsilon)O_{20} , \\ &O^m_{30} = O_{40} . \end{aligned} \tag{4.49}$$

The renormalized operators are related by:

$$\begin{aligned} O^m_3(\mu) &= O_4(\mu) \\ &+ \left[\tilde{Z}^{-1}Z_m^{-1}Z^m - (1-\varepsilon)\tilde{Z}\left(\bar{Z}^m_a + (3-2\varepsilon)\bar{Z}^m_b\right)\right] O_2(\mu) . \end{aligned} \tag{4.50}$$

Therefore,

$$\tilde{Z}^{-1}Z_m^{-1}Z^m - (1-\varepsilon)\tilde{Z}\left(\bar{Z}^m_a + (3-2\varepsilon)\bar{Z}^m_b\right)$$

must be finite at $\varepsilon \to 0$. This allows one to reconstruct γ^m in (4.48) from $\gamma^m_{a,b}$ in (4.45):

$$\begin{aligned} \gamma^m &= -\gamma^m_a - 3\gamma^m_b + \Delta\gamma^m , \\ \Delta\gamma^m &= \left[\tfrac{1}{2}\left(\gamma_{m0} - 2\beta_0\right)\left(\gamma^m_{a0} + 5\gamma^m_{b0}\right) - \tfrac{1}{3}\gamma_{a0}\gamma^m_{a0}\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3) . \end{aligned} \tag{4.51}$$

With

$$\begin{aligned} \gamma^m &= 8C_F\frac{\alpha_s}{4\pi} + C_F\left[\left(\frac{128}{9}\pi^2 + \frac{32}{3}\right)C_F\right. \\ &\quad \left. + \left(-\frac{32}{9}\pi^2 + \frac{548}{9}\right)C_A - \frac{160}{9}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \end{aligned} \tag{4.52}$$

The finite part, at the next-to-leading order, is

$$O_3^m(\mu) = O_4(\mu) + \tfrac{1}{2}\left(\gamma_{a0}^m + 5\gamma_{b0}^m\right)\frac{\alpha_s(\mu)}{4\pi}O_2(\mu)\,. \tag{4.53}$$

The Wilson coefficients $B_i(\mu',\mu)$ have to be determined. From the bilocal insertion (4.20) and the reparametrization relation (4.22) one obtains,

$$\begin{aligned} &B_1^1 = B_3^1 = C_1\,, \quad B_4^1 = C_m C_1\,,\\ &B_1^{\not{v}} = C_{\not{v}} - 2C_{\gamma_\perp}\,, \quad B_3^{\not{v}} = C_{\not{v}}\,, \quad B_4^{\not{v}} = C_m C_{\not{v}}\,. \end{aligned} \tag{4.54}$$

They follow the renormalization group equation:

$$\frac{\partial B^\Gamma(\mu',\mu)}{\partial \log\mu} = \gamma^T(\alpha_s(\mu))B^\Gamma(\mu',\mu) \tag{4.55}$$

The unknown coefficients $B_2^\Gamma(\mu',\mu)$ for $\Gamma = 1, \not{v}$ are obtained by solving the renormalization-group equations

$$\frac{\partial B_2^\Gamma}{\partial \log\mu} = \tilde{\gamma}B_2^\Gamma + \gamma^k B_3^\Gamma + \gamma^m B_4^\Gamma\,, \tag{4.56}$$

with the initial conditions $B_2^\Gamma(m,m)$ obtained by matching at $\mu' = \mu = m$. The ratio $B_2^\Gamma(\mu',\mu)/C_\Gamma(\mu',\mu)$ does not depend on μ':

$$\begin{aligned} \frac{B_2^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} &= \frac{\hat{B}_2^\Gamma}{\hat{C}_\Gamma} - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)}\frac{d\alpha_s}{\alpha_s}\\ &- \hat{C}_m \alpha_s(\mu_0)^{\frac{\gamma_{m0}}{2\beta_0}} \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma^m(\alpha_s)}{2\beta(\alpha_s)} K_{-\gamma_m}(\alpha_s)\alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}} \frac{d\alpha_s}{\alpha_s} \end{aligned} \tag{4.57}$$

(see (2.23)). The renormalization-group invariants $\hat{B}_2^\Gamma$ start at one loop: $\hat{B}_2^\Gamma = b_{21}^\Gamma \alpha_s(\mu_0)/(4\pi) + \cdots$.

$B_2^\Gamma(m,m)$ for $\Gamma = 1, \not{v}$ with the one-loop accuracy are obtained by writing down the sum in (4.19) via the bare operators:

$$\begin{aligned} &C_F\frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)b_\Gamma O_{20} \pm O_{10} + O_{30} + O_{40}\\ &= \left(C_F b_\Gamma - \frac{\gamma_0^k + \gamma_0^m}{2}\right)\frac{\alpha_s(m)}{4\pi\varepsilon}O_2(m) + (\text{other operators})\,. \end{aligned}$$

Taking $\gamma_0^k + \gamma_0^m = 0$ into account, both b_Γ should vanish at $\varepsilon = 0$. The $\mathcal{O}(\varepsilon)$ terms of (4.27) give [125–127]

$$B_2^1(m,m) = 8C_F\frac{\alpha_s(m)}{4\pi} + \cdots\,, \quad B_2^{\not{v}}(m,m) = 12C_F\frac{\alpha_s(m)}{4\pi} + \cdots\,,$$

and

$$\hat{B}_2^1 = 8C_F\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\,, \quad \hat{B}_2^{\not{v}} = 12C_F\frac{\alpha_s(\mu_0)}{4\pi} + \cdots \tag{4.58}$$

(using $\gamma_0^k + \gamma_0^m = 0$).

An exact relation between $\hat{B}_2^1$ and $\hat{B}_2^{\not v}$ can be derived. The QCD vector current and the scalar one are related by the equations of motion:

$$i\partial_\alpha j_0^\alpha = i\partial_\alpha j^\alpha = m_0 j_0 = m(\mu')j(\mu')\,, \tag{4.59}$$

where

$$m(\mu') = \hat{m}\left(\frac{\alpha_s'(\mu')}{\alpha_s'(\mu_0)}\right)^{-\frac{\gamma_{10}'}{2\beta_0'}} K'_{-\gamma_1'}(\alpha_s'(\mu'))$$

is the n_f-flavour $\overline{\mathrm{MS}}$ running mass, and (4.5) $\gamma_1' = -6C_F\alpha_s/(4\pi) + \cdots$ is minus the mass anomalous dimension. $j^\alpha = (j\cdot v)v^\alpha + j_\perp^\alpha$ is separated and substitute the expansions (4.19) with (4.47), (4.54). The matrix element of (4.59) from the heavy quark with momentum mv to the on-shell light quark with momentum p reads

$$\begin{aligned} & mC_{\not v}(\mu',\mu)\left\{1+\frac{1}{2m}\left[\left(\frac{B_2^{\not v}(\mu',\mu)}{C_{\not v}(\mu',\mu)} + 2\left(\frac{C_{\gamma_\perp}(\mu',\mu)}{C_{\not v}(\mu',\mu)} - 1\right)\right)p\cdot v + r\right]\right\} \\ & = m(\mu')C_1(\mu',\mu)\left\{1+\frac{1}{2m}\left[\frac{B_2^1(\mu',\mu)}{C_1(\mu',\mu)}p\cdot v + r\right]\right\}\,, \end{aligned}$$

where

$$r = \frac{<q|i\int dx\, T\left\{\tilde{\jmath}(\mu), (O_k + C_m(\mu)O_m(\mu))_x\right\}|Q> - p_\alpha <q|\tilde{\jmath}^\alpha(\mu)|Q>}{<q|\tilde{\jmath}(\mu)|Q>}\,,$$

$\tilde{\jmath}^\alpha = \bar{q}\gamma^\alpha h_v$. At leading order in $1/m$, this yields [50]

$$\frac{m}{m(\mu')} = \frac{C_1(\mu',\mu)}{C_{\not v}(\mu',\mu)} \quad \text{or} \quad \frac{m}{\hat{m}} = \frac{\hat{C}_1}{\hat{C}_{\not v}}\,. \tag{4.60}$$

At first order, yields,

$$\begin{aligned} & \frac{B_2^1(\mu',\mu)}{C_1(\mu',\mu)} - \frac{B_2^{\not v}(\mu',\mu)}{C_{\not v}(\mu',\mu)} = 2\left(\frac{C_{\gamma_\perp}(\mu',\mu)}{C_{\not v}(\mu',\mu)} - 1\right) \\ \text{or} \quad & \frac{\hat{B}_2^1}{\hat{C}_1} - \frac{\hat{B}_2^{\not v}}{\hat{C}_{\not v}} = 2\left(\frac{\hat{C}_{\gamma_\perp}}{\hat{C}_{\not v}} - 1\right)\,. \end{aligned} \tag{4.61}$$

Note that (4.28) is just the one-loop case of this general result. The one-loop results (4.58), of course, satisfy this requirement.

As discussed above, these results do not change when replacing $\bar{q} \to \bar{q}\gamma_5^{\mathrm{AC}}$. Now the leading term is $\tilde{\jmath} = \bar{q}\gamma_5^{\mathrm{AC}}h_v$, and the definitions of O_i (4.47) are changed accordingly. Taking matrix elements between B meson states:

$$\begin{aligned} & <0|\left(\bar{q}\gamma_5^{\mathrm{AC}}Q\right)_{\mu'}|B> = -im_B f_B^P(\mu')\,, \\ & <0|\bar{q}\gamma_5^{\mathrm{AC}}\gamma^\alpha Q|B> = -if_B p_B^\alpha\,, \end{aligned} \tag{4.62}$$

where

$$f_B^P(\mu') = \hat{f}_B^P\left(\frac{\alpha_s'(\mu')}{\alpha_s'(\mu_0)}\right)^{\frac{\gamma_{10}'}{2\beta_0'}} K'_{\gamma_1'}(\alpha_s'(\mu'))\,. \tag{4.63}$$

The HQET matrix element of the leading order current [111]:

$$\langle 0|\tilde{\jmath}(\mu)|B\rangle = -i\sqrt{m_B}F(\mu)\,, \tag{4.64}$$

where $|B\rangle$ is understood as the effective state and

$$F(\mu) = \hat{F}\left(\frac{\alpha_s(\mu)}{4\pi}\right)^{\frac{\tilde{\gamma}_0}{2\beta_0}} K_{\tilde{\gamma}}(\alpha_s(\mu))\,, \tag{4.65}$$

with $\hat{F}$, μ-independent, is thus just a (non-perturbative) number times $\Lambda_{\overline{\rm MS}}^{3/2}$. For $O_{1,2}$ the full derivative pick the difference of the momenta of the states [111]:

$$\langle 0|O_1(\mu)|B\rangle = -\langle 0|O_2|B\rangle = -i\sqrt{m_B}\bar{\Lambda}F(\mu)\,, \tag{4.66}$$

where $\bar{\Lambda} = m_B - m$ is the B-meson residual energy. O_3 following [111] is defined by:

$$\langle 0|O_3(\mu)|B\rangle = -i\sqrt{m_B}F(\mu)G_k(\mu)\,, \tag{4.67}$$

However, the formulae of [111] only hold for O_4 at leading order. Following [111]

$$\langle 0|O_3^m(\mu)|M\rangle = \frac{1}{12}F(\mu)G(\mu)\mathrm{Tr}[\Gamma P_+\sigma_{\mu\nu}\mathcal{M}\sigma^{\mu\nu}]. \tag{4.68}$$

Defining

$$\langle 0|O_4(\mu)|B\rangle = -i\sqrt{m_B}F(\mu)G_m(\mu), \tag{4.69}$$

at the next to leading order, one obtains:

$$G_m(\mu) = G(\mu) + R_m^B(\alpha_s(\mu))\bar{\Lambda}\,, \qquad R_m^B = \frac{1}{2}\left(\gamma_{10}^m + 5\gamma_{20}^m\right)\frac{\alpha_s(\mu)}{4\pi} \tag{4.70}$$

The hadronic parameters $G_{k,m}(\mu)$ obey the renormalization-group equations

$$\begin{aligned}
&\frac{dG_k(\mu)}{d\log\mu} = \gamma^k(\alpha_s(\mu))\bar{\Lambda}\,,\\
&\frac{dG_m(\mu)}{d\log\mu} + \gamma_m(\alpha_s(\mu))G_m(\mu) = \gamma^m(\alpha_s(\mu))\bar{\Lambda}\,.
\end{aligned} \tag{4.71}$$

Their solution is

$$\begin{aligned}
G_k(\mu) &= \hat{G}_k - \bar{\Lambda}\left[\frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu)}{4\pi} + \int_0^{\alpha_s(\mu)}\left(\frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0^k}{2\beta_0}\right)\frac{d\alpha_s}{\alpha_s}\right],\\
C_m(\mu)G_m(\mu) &= \hat{C}_m\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\\
&\times\left[\hat{G}_m - \bar{\Lambda}\int_0^{\alpha_s(\mu)}\frac{\gamma^m(\alpha_s)}{2\beta(\alpha_s)}K_{-\gamma_m}(\alpha_s)\left(\frac{\alpha_s}{4\pi}\right)^{-\frac{\gamma_{m0}}{2\beta_0}}\frac{d\alpha_s}{\alpha_s}\right],
\end{aligned} \tag{4.72}$$

where $\hat{G}_k$ and $\hat{G}_m$ are again μ-independent and thus are just some (non-perturbative) numbers times $\Lambda_{\overline{\rm MS}}$.

Taking the matrix element of (4.19), it results in

$$\begin{Bmatrix} f_B^P(\mu') \\ f_B \end{Bmatrix} = \frac{C_\Gamma(\mu',\mu)F(\mu)}{\sqrt{m_B}} \times \left[1 + \frac{1}{2m}\left(C_\Lambda^\Gamma(\mu)\bar\Lambda + G_k(\mu) + C_m(\mu)G_m(\mu)\right)\right], \tag{4.73}$$

where $\Gamma = 1$, $\not{v}$, and

$$C_\Lambda^1(\mu) = 1 - \frac{B_2^1(\mu',\mu)}{C_1(\mu',\mu)}, \quad C_\Lambda^{\not{v}}(\mu) = 1 - 2\frac{C_{\gamma_\perp}(\mu',\mu)}{C_{\not{v}}(\mu',\mu)} - \frac{B_2^{\not{v}}(\mu',\mu)}{C_{\not{v}}(\mu',\mu)}.$$

Substituting the solutions of the renormalization-group equations, one arrives at the explicitly μ-independent expressions

$$\begin{Bmatrix} \hat f_B^P \\ f_B \end{Bmatrix} = \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\tilde\gamma_0}{2\beta_0}} \frac{\hat C_\Gamma \hat F}{\sqrt{m_B}} \times \left[1 + \frac{1}{2m}\left(\hat C_\Lambda^\Gamma\bar\Lambda + \hat G_k + \hat C_m \hat G_m \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\right)\right], \tag{4.74}$$

where

$$\hat C_\Lambda^\Gamma = 1 - 2\begin{Bmatrix} 0 \\ \hat C_{\gamma_\perp}/\hat C_{\not{v}} \end{Bmatrix} - \frac{\hat B_2^\Gamma}{\hat C_\Gamma} - \frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu_0)}{4\pi} - \int_0^{\alpha_s(\mu_0)}\left(\frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0^k}{2\beta_0}\right)\frac{d\alpha_s}{\alpha_s} - \hat C_m \alpha_s(\mu_0)^{\frac{\gamma_{m0}}{2\beta_0}} \int_0^{\alpha_s(\mu_0)} \frac{\gamma^m(\alpha_s)}{2\beta(\alpha_s)} K_{-\gamma_m}(\alpha_s)\alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}}\frac{d\alpha_s}{\alpha_s}.$$

At next-to-leading order,

$$\begin{Bmatrix} \hat f_B^P \\ f_B \end{Bmatrix} = \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\tilde\gamma_0}{2\beta_0}} \frac{\hat C_\Gamma \hat F}{\sqrt{m_B}} \Bigg\{1 + \frac{1}{2m}\Bigg[\left(-\frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu_0)}{4\pi} \pm 1 + \frac{\gamma_0^m}{\gamma_{m0}} + c_{\Lambda 1}^\Gamma \frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right)\bar\Lambda + \hat G_k + \hat G_m \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left(1 + c_{m1}\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right)\Bigg]\Bigg\}, \tag{4.75}$$

where

$$c_{\Lambda 1}^\Gamma = (1 \mp 1)\left(c_1^{\not{v}} - c_1^{\gamma_\perp}\right) - b_{21}^\Gamma + \frac{\gamma_0^m}{\gamma_{m0}}c_{m1} - \frac{\gamma_1^k + \gamma_1^m}{2\beta_0} + \frac{\beta_1\left(\gamma_0^k + \gamma_0^m\right)}{2\beta_0^2} + \frac{\gamma_{m0}\gamma_1^m - \gamma_0^m\gamma_{m1}}{2\beta_0\left(\gamma_{m0} - 2\beta_0\right)}$$

(here $\gamma_0^k + \gamma_0^m = 0$).

The matrix element of (4.59) is

$$\frac{f_B^P(\mu')}{f_B} = \frac{m_B}{m(\mu')} \quad \text{or} \quad \frac{\hat{f}_B^P}{f_B} = \frac{m_B}{\hat{m}} \,. \tag{4.76}$$

Substituting (4.73) (or (4.74)) and using the relation (4.61), one obtains

$$\frac{f_B^P(\mu')}{f_B} = \frac{C_1(\mu',\mu)}{C_{\not{v}}(\mu',\mu)}\left(1+\frac{\bar\Lambda}{m}\right) \quad \text{or} \quad \frac{\hat{f}_B^P}{f_B} = \frac{\hat{C}_1}{\hat{C}_{\not{v}}}\left(1+\frac{\bar\Lambda}{m}\right) . \tag{4.77}$$

The ratio of the quark masses is given by (4.60). Naturally, it contains no $1/m$ corrections with B-meson hadronic parameters; it is just a series in $\alpha_s(\mu_0)$, see the Appendix A. In the next section the Spin 1 currents will be studied.

4.4 Spin-1 currents

In the case of the currents with $\Gamma = \gamma_\perp^\alpha$, $\gamma_\perp^\alpha \not{v}$, the leading term in the expansion (4.19) contains $\tilde{\jmath}^\alpha = \bar{q}\gamma_\perp^\alpha h_v$, and the $1/m$ correction; six operators

$$\begin{aligned}
O_1^\alpha &= \bar{q} i D^\alpha h_v \,, \\
O_2^\alpha &= \bar{q} i \not{D} \gamma_\perp^\alpha h_v = i\partial_\beta \left(\bar{q}\gamma^\beta\gamma_\perp^\alpha h_v\right) \,, \\
O_3^\alpha &= \bar{q}\left(-i\overleftarrow{D}_\perp^\alpha\right) h_v \,, \\
O_4^\alpha &= \bar{q}\left(-iv\cdot\overleftarrow{D}\right)\gamma_\perp^\alpha h_v = -iv\cdot\partial\left(\bar{q}\gamma_\perp^\alpha h_v\right) \,, \\
O_5^\alpha &= i\int dx\, T\left\{\tilde{\jmath}^\alpha(0), O_k(x)\right\} \,, \\
O_6^\alpha &= i\int dx\, T\left\{\tilde{\jmath}^\alpha(0), O_m(x)\right\} \,.
\end{aligned} \tag{4.78}$$

From the insertion of the Lagrangian (4.20) and reparametrization invariance (4.22),

$$\begin{aligned}
&B_1^{\gamma_\perp}(\mu) = 2C_{\not{v}}(\mu)\,, \quad -B_2^{\gamma_\perp}(\mu) = B_5^{\gamma_\perp}(\mu) = C_{\gamma_\perp}(\mu)\,, \\
&B_6^{\gamma_\perp}(\mu) = C_m(\mu) C_{\gamma_\perp}(\mu)\,, \\
&-\tfrac{1}{2}B_1^{\gamma_\perp\not{v}}(\mu) = B_2^{\gamma_\perp\not{v}}(\mu) = B_5^{\gamma_\perp\not{v}}(\mu) = C_{\gamma_\perp\not{v}}(\mu)\,, \\
&B_6^{\gamma_\perp\not{v}}(\mu) = C_m(\mu) C_{\gamma_\perp\not{v}}(\mu)\,.
\end{aligned} \tag{4.79}$$

The renormalization constant and anomalous dimension matrix of the dimension-4 operators (4.78) have the structure [126],

$$Z = \tilde{Z} + \begin{pmatrix} 0 & 0 & Z_a & Z_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & Z_a & Z_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & Z^k & 0 & 0 \\ 0 & 0 & Z_1^m & Z_2^m & 0 & Z_m \end{pmatrix} , \quad \gamma = \tilde{\gamma} + \begin{pmatrix} 0 & 0 & \gamma_a & \gamma_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \gamma_a & \gamma_b & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \gamma^k & 0 & 0 \\ 0 & 0 & \gamma_1^m & \gamma_2^m & 0 & \gamma_m \end{pmatrix} . \tag{4.80}$$

The operators $O^\alpha_{2,4}$ are renormalized multiplicatively with $\tilde{Z}$, which determines the second and the fourth rows. The same holds for

$$O^\alpha_{10} - O^\alpha_{30} = i\partial^\alpha_\perp(\bar{q}_o h_{v0}) = \tilde{Z}(O^\alpha_1 - O^\alpha_3), \tag{4.81}$$

and hence the first and the third rows coincide. Furthermore, the form of $B_{1,2,5,6}(\mu)$ (see (4.79)) fixes the columns 1, 2, 5, 6. From Sect. 4.2 for $\Gamma = 1$, $O^\Gamma_1 = O^\alpha_1$ and hence receives counterterms from $O^\Gamma_2 = O^\alpha_3$ and $O^\Gamma_3 = O^\alpha_4$. The bilocal operator $O^\alpha_5 = O^k_2$ with $\Gamma = \gamma^\alpha_\perp$ mixes with $O^\alpha_4 = O^k_1$. Finally, $O^\alpha_6 = O^m_3$ mixes with $O^\alpha_{3,4}$.

One can reconstruct the anomalous dimension $\gamma^m_{1,2}$ from the known $\gamma^m_{a,b}$. For $\Gamma = \gamma^\alpha_\perp$, the operators (4.44) are related to (4.78):

$$\begin{aligned} &O^m_{10} = (1-\varepsilon)(1+2\varepsilon)O^\alpha_{40}\,, \quad O^m_{20} = (1-2\varepsilon)O^\alpha_{30} + \varepsilon O^\alpha_{40}\,, \\ &O^m_{30} = O^\alpha_{60}\,. \end{aligned} \tag{4.82}$$

From the renormalized operators

$$\begin{aligned} O^m_3(\mu) &= O^\alpha_6(\mu) \\ &+ \left[\tilde{Z}^{-1} Z_m^{-1} Z^m_1 + (1-2\varepsilon)(\tilde{Z} + Z_a)\bar{Z}^m_a\right] O^\alpha_3(\mu) \\ &+ \left[\tilde{Z}^{-1} Z_m^{-1} Z^m_2 + \left((1-2\varepsilon)Z_b + \varepsilon\tilde{Z}\right)\bar{Z}^m_a + (1-\varepsilon)(1+2\varepsilon)\tilde{Z}\bar{Z}^m_b\right] O^\alpha_4(\mu)\,. \end{aligned} \tag{4.83}$$

$$\tilde{Z}^{-1} Z_m^{-1} Z^m_1 + (1-2\varepsilon)(\tilde{Z} + Z_a)\bar{Z}^m_a$$

and

$$\tilde{Z}^{-1} Z_m^{-1} Z^m_2 + \left((1-2\varepsilon)Z_b + \varepsilon\tilde{Z}\right)\bar{Z}^m_a + (1-\varepsilon)(1+2\varepsilon)\tilde{Z}\bar{Z}^m_b$$

must be finite at $\varepsilon \to 0$. This allows one to reconstruct $\gamma^m_{1,2}$ in (4.80) from $\gamma^m_{a,b}$ in (4.45):

$$\begin{aligned} &\gamma^m_1 = \gamma^m_a + \Delta\gamma^m_1\,, \quad \gamma^m_2 = \gamma^m_b + \Delta\gamma^m_2\,, \\ &\Delta\gamma^m_1 = \gamma^m_{a0}\left(\gamma_{a0} - \gamma_{m0} + 2\beta_0\right)\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\,, \\ &\Delta\gamma^m_2 = \left[\tfrac{1}{2}\left(\gamma_{m0} - 2\beta_0\right)\left(\gamma^m_{a0} + \gamma^m_{b0}\right) - \tfrac{1}{3}\gamma_{a0}\gamma^m_{a0}\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\,. \end{aligned} \tag{4.84}$$

The anomalous dimensions $\gamma^m_{1,2}$ has been calculated in [113] to two-loop accuracy (they are called $\gamma^{\text{mag}}_{3,1}$ in [113]):

$$\begin{aligned} \gamma^m_1 &= -2C_F\frac{\alpha_s}{4\pi} + C_F\left[\left(-\frac{8}{9}\pi^2 + \frac{4}{3}\right)C_F \right. \\ &\quad \left. + \left(\frac{2}{9}\pi^2 - \frac{206}{9}\right)C_A + \frac{52}{9}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \\ \gamma^m_2 &= -2C_F\frac{\alpha_s}{4\pi} + C_F\left[\left(-\frac{40}{9}\pi^2 - \frac{10}{3}\right)C_F \right. \\ &\quad \left. + \left(\frac{10}{9}\pi^2 + \frac{46}{9}\right)C_A - \frac{44}{9}T_F n_l\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \cdots \end{aligned} \tag{4.85}$$

which satisfies (4.84). The anomalous dimension γ^m (4.52) is, from (4.51) and (4.84),

$$\begin{aligned}
\gamma^m &= -\gamma_1^m - 3\gamma_2^m + \Delta\gamma\,, \\
\Delta\gamma &= \left[(\gamma_{m0} - 2\beta_0)\left(\gamma_{10}^m + 4\gamma_{20}^m\right) - \tfrac{1}{3}\gamma_{a0}\gamma_{10}^m\right]\left(\frac{\alpha_s}{4\pi}\right)^2 + \mathcal{O}(\alpha_s^3)\,.
\end{aligned} \tag{4.86}$$

The finite part, at the next-to-leading order, is

$$O_3^m(\mu) = O_6^\alpha(\mu) - \gamma_{a0}^m \frac{\alpha_s(\mu)}{4\pi} O_3^\alpha(\mu) + \tfrac{1}{2}\left(\gamma_{a0}^m + \gamma_{b0}^m\right)\frac{\alpha_s(\mu)}{4\pi} O_4^\alpha(\mu)\,. \tag{4.87}$$

The unknown coefficients $B_{3,4}^\Gamma(\mu',\mu)$ for $\Gamma = \gamma_\perp,\ \gamma_\perp \not{v}$ are obtained by solving the renormalization-group equations

$$\begin{aligned}
\frac{\partial B_3^\Gamma}{\partial \log\mu} &= (\tilde\gamma + \gamma_a)\, B_3^\Gamma + \gamma_a B_1^\Gamma + \gamma_1^m B_6^\Gamma\,, \\
\frac{\partial B_4^\Gamma}{\partial \log\mu} &= \tilde\gamma B_4^\Gamma + \gamma_b\left(B_1^\Gamma + B_3^\Gamma\right) + \gamma^k B_5^\Gamma + \gamma_2^m B_6^\Gamma
\end{aligned} \tag{4.88}$$

(where (4.5) $\gamma'_{\gamma_\perp \not{v}} = 2C_F\alpha_s/(4\pi) + \cdots$), with the initial conditions $B_{3,4}^\Gamma(m,m)$ obtained by matching. The ratios $B_i^\Gamma(\mu',\mu)/C_\Gamma(\mu',\mu)$ do not depend on μ':

$$\begin{aligned}
\frac{B_3^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} &= \left(\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}\right)^{-\frac{\gamma_{a0}}{2\beta_0}} K_{-\gamma_a}(\alpha_s(\mu)) \Bigg[\frac{\hat B_3^\Gamma}{\hat C_\Gamma} \\
&\quad - \left\{\begin{matrix}\hat C_{\not{v}}/\hat C_{\gamma_\perp} \\ -1\end{matrix}\right\} \alpha_s(\mu_0)^{-\frac{\gamma_{a0}}{2\beta_0}} \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma_a(\alpha_s)}{\beta(\alpha_s)} K_{\gamma_a}(\alpha_s)\alpha_s^{\frac{\gamma_{a0}}{2\beta_0}} \frac{d\alpha_s}{\alpha_s} \\
&\quad - \hat C_m \alpha_s(\mu_0)^{\frac{\gamma_{m0}-\gamma_{a0}}{2\beta_0}} \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma_1^m(\alpha_s)}{2\beta(\alpha_s)} K_{\gamma_a-\gamma_m}(\alpha_s)\alpha_s^{\frac{\gamma_{a0}-\gamma_{m0}}{2\beta_0}} \frac{d\alpha_s}{\alpha_s}\Bigg]\,,
\end{aligned} \tag{4.89}$$

$$\begin{aligned}
\frac{B_4^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} &= \frac{\hat B_4^\Gamma}{\hat C_\Gamma} - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma_b(\alpha_s)}{2\beta(\alpha_s)}\left[\left.\frac{B_3^\Gamma}{C_\Gamma}\right|_{\alpha_s} + 2\left\{\begin{matrix}\hat C_{\not{v}}/\hat C_{\gamma_\perp} \\ -1\end{matrix}\right\}\right]\frac{d\alpha_s}{\alpha_s} \\
&\quad - \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)}\frac{d\alpha_s}{\alpha_s} \\
&\quad - \hat C_m \alpha_s(\mu_0)^{\frac{\gamma_{m0}}{2\beta_0}} \int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)} \frac{\gamma_2^m(\alpha_s)}{2\beta(\alpha_s)} K_{-\gamma_m}(\alpha_s)\alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}}\frac{d\alpha_s}{\alpha_s}
\end{aligned} \tag{4.90}$$

(in the last formula, the running $B_3^\Gamma(\mu',\mu)/C_\Gamma(\mu',\mu)$ corresponding to the integration variable α_s is understood).

$B_{3,4}^\Gamma(m,m)$ for $\Gamma = \gamma_\perp,\ \gamma_\perp\not{v}$ is found to one-loop accuracy by writing down the sum in (4.19) via the bare operators:

$$\begin{aligned}
&C_F \frac{g_0^2 m^{-2\varepsilon}}{(4\pi)^{d/2}}\Gamma(\varepsilon)\left(b_{\Gamma,2}O_{30}^\alpha + b_{\Gamma,1}O_{40}^\alpha\right) \pm O_{10} \mp O_{20} + O_{50} + O_{60} \\
&= \frac{\alpha_s(m)}{4\pi\varepsilon}\Bigg[\left(C_F b_{\Gamma,2} \mp \gamma_{a0} - \frac{\gamma_{10}^m}{2}\right) O_3^\alpha(m) \\
&\quad + \left(C_F b_{\Gamma,1} \mp \gamma_{b0} - \frac{\gamma_0^k + \gamma_{20}^m}{2}\right) O_4^\alpha(m) + (\text{other operators})\Bigg]\,.
\end{aligned}$$

The values of $b_{\Gamma,i}$ at $\varepsilon = 0$ have to cancel these anomalous dimensions. The $\mathcal{O}(\varepsilon)$ terms of (4.27) give

$$B_3^{\gamma_\perp}(m,m) = 4C_F\frac{\alpha_s(m)}{4\pi} + \cdots\,,\quad B_4^{\gamma_\perp}(m,m) = -4C_F\frac{\alpha_s(m)}{4\pi} + \cdots\,,$$
$$B_3^{\gamma_\perp \not{v}}(m,m) = 2C_F\frac{\alpha_s(m)}{4\pi} + \cdots\,,\quad B_4^{\gamma_\perp \not{v}}(m,m) = -6C_F\frac{\alpha_s(m)}{4\pi} + \cdots$$

The results for $\Gamma = \gamma_\perp$ were obtained in [125, 126]; those for $\Gamma = \gamma_\perp\not{v}$ are new. Using the one-loop anomalous dimensions, the related invariant renormalization coefficients are:

$$\hat{B}_3^{\gamma_\perp} = \frac{2}{3}C_F\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\,,\quad \hat{B}_4^{\gamma_\perp} = 6C_F\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\,,$$
$$\hat{B}_3^{\gamma_\perp\not{v}} = \frac{26}{3}C_F\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\,,\quad \hat{B}_4^{\gamma_\perp\not{v}} = \frac{2}{3}C_F\frac{\alpha_s(\mu_0)}{4\pi} + \cdots \tag{4.91}$$

Taking matrix elements

$$\begin{aligned} &<0|\bar{q}\gamma^\alpha Q|B^*> = m_{B^*} f_{B^*} e^\alpha\,, \\ &<0|\left(\bar{q}\sigma^{\alpha\beta}Q\right)_{\mu'}|B^*> = i f_{B^*}^T(\mu')\left(e^\alpha p_{B^*}^\beta - e^\beta p_{B^*}^\alpha\right)\,, \\ &f_{B^*}^T(\mu') = \hat{f}_{B^*}^T\left(\frac{\alpha_s'(\mu')}{\alpha_s'(\mu_0)}\right)^{\frac{\gamma_{\sigma 0}'}{2\beta_0'}} K_{\gamma_\sigma'}'(\alpha_s'(\mu'))\,. \end{aligned} \tag{4.92}$$

The HQET matrix elements are [111]

$$\begin{aligned} &<0|\tilde{\jmath}^\alpha(\mu)|B^*> = \sqrt{m_{B^*}}F(\mu)e^\alpha\,, \\ &<0|O_2^\alpha(\mu)|B> = <0|O_4^\alpha(\mu)|B> = -\sqrt{m_{B^*}}\bar{\Lambda}F(\mu)e^\alpha\,. \end{aligned} \tag{4.93}$$

The matrix elements of O_1^α and O_3^α are equal, due to (4.81). However, the formulae [111] for these matrix elements hold only at the leading order. Let us define

$$<0|O_1^\alpha(\mu)|B> = <0|O_3^\alpha(\mu)|B> = -\frac{1}{3}\sqrt{m_{B^*}}\bar{\Lambda}F(\mu)R(\alpha_s(\mu))e^\alpha\,, \tag{4.94}$$

where $R = 1 + \mathcal{O}(\alpha_s)$. It obeys the renormalization-group equation

$$\frac{dR}{d\log\mu} + \gamma_a R + 3\gamma_b = 0\,. \tag{4.95}$$

Following [111], it is defined

$$<0|O_1^\Gamma(\mu)|M> = \frac{F_2(\mu)}{2}\,\mathrm{Tr}\,\gamma_\perp^\alpha\Gamma\mathcal{M}\,, \tag{4.96}$$

where $M = B$ or B^*, and $\mathcal{M}$ is the corresponding Dirac structure. Now, using $\Gamma = \gamma_\alpha\Gamma'$

$$<0|O_1^\Gamma(\mu)|M> = \frac{3}{2}F_2(\mu)\,\mathrm{Tr}\,\Gamma'\mathcal{M}\,.$$

Taking into account (4.42) and

$$<0|O_1'(\mu)|M> = <0|O_2'(\mu)|M> = -\frac{1}{2}\bar{\Lambda}F(\mu)\,\mathrm{Tr}\,\Gamma'\mathcal{M}\,,$$

at the next-to-leading order, one obtains:

$$F_2(\mu) = -\frac{1}{3}\bar{\Lambda}F(\mu)R(\alpha_s(\mu))\,, \quad R(\alpha_s) = 1 + \frac{1}{3}\gamma_{a0}\frac{\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2)\,. \tag{4.97}$$

The general result for $R(\alpha_s)$ can be derived by solving (4.95) and requiring the absence of fractional powers of α_s (or by requiring that (4.97) is reproduced):

$$R(\alpha_s) = K_{\gamma_a}(\alpha_s)\left[1 + \alpha_s^{\frac{\gamma_{a0}}{2\beta_0}}\int_0^{\alpha_s}\left(\frac{3\gamma_b(\alpha_s)}{2\beta(\alpha_s)}K_{-\gamma_a}(\alpha_s) + \frac{\gamma_{a0}}{2\beta_0}\right)\alpha_s^{-\frac{\gamma_{a0}}{2\beta_0}}\frac{d\alpha_s}{\alpha_s}\right]\,. \tag{4.98}$$

The matrix element of O_5^α is [111]

$$<0|O_5^\alpha(\mu)|B^*> = \sqrt{m_{B^*}}F(\mu)G_k(\mu)e^\alpha\,. \tag{4.99}$$

However, the formulae [111] for the matrix element of O_6^α hold only at the leading order. Following [111], one can define

$$<0|O_3^m(\mu)|M> = \frac{1}{12}F(\mu)G(\mu)\,\mathrm{Tr}\,\Gamma\frac{1+\not{v}}{2}\sigma_{\mu\nu}\mathcal{M}\sigma^{\mu\nu}\,. \tag{4.100}$$

For B^*, defining

$$<0|O_6^\alpha|B^*> = -\frac{1}{3}\left[G_m(\mu) + R_m(\alpha_s(\mu))\bar{\Lambda}\right]\sqrt{m_{B^*}}F(\mu)e^\alpha\,, \tag{4.101}$$

at the next-to-leading order (4.87), one obtains:

$$R_m(\alpha_s) = -\left(\gamma_{10}^m + 4\gamma_{20}^m\right)\frac{\alpha_s}{4\pi} + \mathcal{O}(\alpha_s^2)\,. \tag{4.102}$$

R_m obeys the renormalization-group equation

$$\frac{d\,R_m}{d\log\mu} + \gamma_m R_m + \gamma^m + \gamma_1^m R + 3\gamma_2^m = 0\,. \tag{4.103}$$

Its solution (which contains no fractional powers of α_s and reproduces (4.102)) is

$$R_m(\alpha_s) = K_{\gamma_m}(\alpha_s)\alpha_s^{\frac{\gamma_{m0}}{2\beta_0}} \tag{4.104}$$
$$\times\int_0^{\alpha_s}\frac{\gamma^m(\alpha_s) + \gamma_1^m(\alpha_s)R(\alpha_s) + 3\gamma_2^m(\alpha_s)}{2\beta(\alpha_s)}K_{-\gamma_m}(\alpha_s)\alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}}\frac{d\alpha_s}{\alpha_s}\,.$$

Taking the matrix element of (4.19), the result for the Spin 1 heavy to light transitions is:

$$\left\{\begin{array}{c} f_{B^*} \\ f_{B^*}^T(\mu') \end{array}\right\} = \frac{C_\Gamma(\mu',\mu)F(\mu)}{\sqrt{m_{B^*}}}$$
$$\times\left[1 + \frac{1}{2m}\left(C_\Lambda^\Gamma(\mu)\bar{\Lambda} + G_k(\mu) - \tfrac{1}{3}C_m(\mu)G_m(\mu)\right)\right]\,, \tag{4.105}$$

where $\Gamma = \gamma_\perp$, $\gamma_\perp \not{v}$, and

$$\begin{aligned} C_\Lambda^\Gamma(\mu) = \pm 1 &- \frac{B_4^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} \\ &- \frac{1}{3}\left(2 \left\{ \begin{array}{c} C_{\not{v}}(\mu',\mu)/C_{\gamma_\perp}(\mu',\mu) \\ -1 \end{array} \right\} + \frac{B_3^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} \right) R(\alpha_s(\mu)) \\ &- \frac{1}{3} C_m(\mu) R_m(\mu) \,. \end{aligned}$$

Substituting the solutions of the renormalization-group equations, one arrives at the μ-independent expressions

$$\begin{aligned} \left\{ \begin{array}{c} f_{B^*} \\ \hat{f}_{B^*}^T \end{array} \right\} = \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^{\frac{\tilde{\gamma}_0}{2\beta_0}} \frac{\hat{C}_\Gamma \hat{F}}{\sqrt{m_{B^*}}} \\ \times \left[1 + \frac{1}{2m} \left(\hat{C}_\Lambda^\Gamma \bar{\Lambda} + \hat{G}_k - \tfrac{1}{3} \hat{C}_m \hat{G}_m \left(\frac{\alpha_s(\mu_0)}{4\pi} \right)^{\frac{\gamma_{m0}}{2\beta_0}} \right) \right] , \end{aligned} \tag{4.106}$$

where

$$\begin{aligned} \hat{C}_\Lambda^\Gamma = \pm 1 &- \frac{B_4^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} \\ &- \frac{1}{3}\left(2 \left\{ \begin{array}{c} C_{\not{v}}(\mu',\mu)/C_{\gamma_\perp}(\mu',\mu) \\ -1 \end{array} \right\} + \frac{B_3^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} \right) R(\alpha_s(\mu)) \\ &- \frac{\gamma_0^k}{2\beta_0} \log \frac{\alpha_s(\mu)}{4\pi} - \int_0^{\alpha_s(\mu)} \left(\frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0^k}{2\beta_0} \right) \frac{d\alpha_s}{\alpha_s} - \frac{1}{3} C_m(\mu) R_m(\alpha_s(\mu)) \\ &+ \frac{1}{3} \hat{C}_m \alpha_s(\mu_0)^{\frac{\gamma_{m0}}{2\beta_0}} \int_0^{\alpha_s(\mu)} \frac{\gamma^m(\alpha_s)}{2\beta(\alpha_s)} K_{-\gamma_m}(\alpha_s) \alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}} \frac{d\alpha_s}{\alpha_s} \,. \end{aligned} \tag{4.107}$$

At first sight, it is not obvious that $\hat{C}_\Lambda^\Gamma$ does not depend on μ. However, differentiating it in $\log\mu$ and taking into account the renormalization-group equations (4.88), (4.95), (4.103), one obtains zero. Therefore, the expression with $\mu = \mu_0$ can be used:

$$\begin{aligned} \hat{C}_\Lambda^\Gamma = \pm 1 - \frac{\hat{B}_4^\Gamma}{\hat{C}_\Gamma} &- \frac{1}{3}\left(2 \left\{ \begin{array}{c} \hat{C}_{\not{v}}/\hat{C}_{\gamma_\perp} \\ -1 \end{array} \right\} + \frac{\hat{B}_3^\Gamma}{\hat{C}_\Gamma} K_{-\gamma_a}(\alpha_s(\mu_0)) \right) R(\alpha_s(\mu_0)) \\ &- \frac{\gamma_0^k}{2\beta_0} \log \frac{\alpha_s(\mu_0)}{4\pi} - \int_0^{\alpha_s(\mu_0)} \left(\frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)} - \frac{\gamma_0^k}{2\beta_0} \right) \frac{d\alpha_s}{\alpha_s} \\ &- \frac{1}{3} \hat{C}_m K_{-\gamma_m}(\alpha_s(\mu_0)) R_m(\alpha_s(\mu_0)) \\ &+ \frac{1}{3} \hat{C}_m \alpha_s(\mu_0)^{\frac{\gamma_{m0}}{2\beta_0}} \int_0^{\alpha_s(\mu_0)} \frac{\gamma^m(\alpha_s)}{2\beta(\alpha_s)} K_{-\gamma_m}(\alpha_s) \alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}} \frac{d\alpha_s}{\alpha_s} \,. \end{aligned} \tag{4.108}$$

Comparing this with (4.107), $B_4^\Gamma(\mu',\mu)/C_\Gamma(\mu',\mu)$ can be rewritten in a form which seems different from (4.90) but is equal to it:

$$\frac{B_4^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} = \frac{\hat{B}_4^\Gamma}{\hat{C}_\Gamma} - \frac{1}{3}\left(2 \left\{ \begin{array}{c} \hat{C}_{\not{v}}/\hat{C}_{\gamma_\perp} \\ -1 \end{array} \right\} + \frac{B_3^\Gamma(\mu',\mu)}{C_\Gamma(\mu',\mu)} \right) R(\alpha_s(\mu))$$

$$
\begin{aligned}
&+\frac{1}{3}\left(2\left\{\begin{array}{c}\hat{C}_{\not v}/\hat{C}_{\gamma_\perp}\\ -1\end{array}\right\}+\frac{\hat{B}_3^\Gamma}{\hat{C}_\Gamma}K_{-\gamma_a}(\alpha_s(\mu_0))\right)R(\alpha_s(\mu_0))\\
&-\frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu)}{\alpha_s(\mu_0)}-\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\left(\frac{\gamma^k(\alpha_s)}{2\beta(\alpha_s)}-\frac{\gamma_0^k}{2\beta_0}\right)\frac{d\alpha_s}{\alpha_s}\\
&-\frac{1}{3}C_m(\mu)R_m(\alpha_s(\mu))+\frac{1}{3}\hat{C}_mK_{-\gamma_m}(\alpha_s(\mu_0))R_m(\alpha_s(\mu_0))\\
&+\frac{1}{3}\hat{C}_m\alpha_s(\mu_0)^{\frac{\gamma_{m0}}{2\beta_0}}\int_{\alpha_s(\mu_0)}^{\alpha_s(\mu)}\frac{\gamma^m(\alpha_s)}{2\beta(\alpha_s)}K_{-\gamma_m}(\alpha_s)\alpha_s^{-\frac{\gamma_{m0}}{2\beta_0}}\frac{d\alpha_s}{\alpha_s}
\end{aligned}
\tag{4.109}
$$

(to convince oneself that this is equivalent to (4.90), one can check that they coincide at $\mu=\mu_0$, and that (4.109) obeys the renormalization-group equation (4.88).)

At next-to-leading order,

$$
\begin{aligned}
\left\{\begin{array}{c}f_{B^*}\\ \hat{f}^T_{B^*}\end{array}\right\}=&\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\tilde{\gamma}_0}{2\beta_0}}\frac{\hat{C}_\Gamma\hat{F}}{\sqrt{m_{B^*}}}\Bigg\{1\\
&+\frac{1}{2m}\Bigg[\left(-\frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu_0)}{4\pi}+\frac{1}{3}\left(\pm1-\frac{\gamma_0^m}{\gamma_{m0}}\right)+c^\Gamma_{\Lambda1}\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\right)\bar{\Lambda}\\
&+\hat{G}_k-\frac{1}{3}\hat{G}_m\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left(1+c_{m1}\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\right)\Bigg]\Bigg\},
\end{aligned}
\tag{4.110}
$$

where

$$
\begin{aligned}
c^\Gamma_{\Lambda1}=&\tfrac{1}{3}(1\pm1)\left(c_1^{\gamma_\perp}-c_1^{\not v}\right)-\tfrac{1}{3}b^\Gamma_{31}-b^\Gamma_{41}-\tfrac{1}{3}\frac{\gamma_0^m}{\gamma_{m0}}c_{m1}\mp\tfrac{2}{9}\gamma_{a0}+\tfrac{1}{3}\gamma_{10}^m+\tfrac{4}{3}\gamma_{20}^m\\
&-\frac{\gamma_1^k-\frac{1}{3}\gamma_1^m}{2\beta_0}+\frac{\beta_1\left(\gamma_0^k-\frac{1}{3}\gamma_0^m\right)}{2\beta_0^2}-\frac{\gamma_{m0}\gamma_1^m-\gamma_0^m\gamma_{m1}}{6\beta_0\left(\gamma_{m0}-2\beta_0\right)}.
\end{aligned}
$$

The ratio $\hat{f}^T_{B^*}/f_{B^*}$ at the leading order in $1/m$ is given by the perturbative series in $\alpha_s(\mu_0)$. (from the result in [50], omitting the $m_c\neq0$ effect, and the three-loop anomalous dimension γ'_σ of the tensor current [114]). This ratio is, from (4.106),

$$
\frac{\hat{f}^T_{B^*}}{f_{B^*}}=\frac{\hat{C}_{\gamma_\perp\not v}}{\hat{C}_{\gamma_\perp}}\left[1-\frac{\bar{\Lambda}}{3m}\left(1+c_{\Lambda1}\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\right)\right],
\tag{4.111}
$$

$$
c_{\Lambda1}=\tfrac{3}{2}\left(c_{\Lambda1}^{\gamma_\perp\not v}-c_{\Lambda1}^{\gamma_\perp}\right)=c_1^{\gamma_\perp}-c_1^{\not v}+\tfrac{1}{2}\left(b_{31}^{\gamma_\perp\not v}-b_{31}^{\gamma_\perp}\right)+\tfrac{3}{2}\left(b_{41}^{\gamma_\perp\not v}-b_{41}^{\gamma_\perp}\right)-\tfrac{2}{3}\gamma_{a0}.
$$

Recall that meson decay constants are fundamental constants of the theory, of non-perturbative character, and non-computable using perturbative methods. Using HQET, at leading order ratios of these quantities are expressed by ratios of perturbative computable Wilson coefficients. Beyond leading order, symmetry-breaking non-perturbative HQET parameter as $\bar{\Lambda}$ enter, which can be computed in the lattice or using sum rules with better accuracy in comparison with meson decay constants themselves.

In the next Chapter, the asymptotic behaviour of the leading Wilson coefficients of the heavy to light currents will be studied giving the chance to study the asymptotic behaviour of the matching coefficient which appear in ratios as f_{B^*}/f_B.

Chapter 5

Asymptotic Behaviour of f_{B^*}/f_B

Contents

The understanding of the structure of the perturbative series has advanced considerably over the recent years, see the review [45]. It became clear that perturbative series are at best asymptotic, not even Borel-summable. Based on an analysis of singularities in the Borel plane, one can obtain the behaviour of the perturbative series for large L, where L is the order of perturbation theory. The nearest singularity determines the leading asymptotic behaviour. Most of the investigations use the large-β_0 limit, whose relation to the real QCD is unclear. At the first order in $1/\beta_0$, singularities in the Borel plane are simple poles. At the higher orders, they become branching points. However, there is an approach [48,49] based on the renormalization group, which yields model-independent results. Singularities in the Borel plane are branching points, whose powers are determined by the relevant anomalous dimensions, but normalization factors cannot be calculated.

Effective field theories make use of the fact that a large scale is present, and physical quantities can be expanded in inverse powers of this large scale. In Heavy Quark Effective Theory (HQET, see the textbook [42]), this scale is the heavy quark mass m. Renormalon singularities in HQET were investigated in [99, 108]. Unlike in QCD, the HQET heavy-quark self-energy has an UV renormalon at positive u, namely $u = 1/2$,which leads to an ambiguity in the residual mass term.

A typical matrix element in the full theory, QCD, is expanded in $1/m$:

$$<j> = C<\tilde{\jmath}> + \frac{1}{2m}\sum B_i<O_i> + \cdots \tag{5.1}$$

(see (4.19)), with short-distance matching coefficients C, B_i,... and long-distance HQET matrix elements $<\tilde{\jmath}>$, $<O_i>$,...

The QCD matrix element $<j>$ contains no renormalon ambiguities, if the operator j has the lowest dimensionality in its channel.[1] In HQET, short- and long-distance contributions are separated. In schemes without strict separation of large and small momenta, such as $\overline{\mathrm{MS}}$, this procedure artificially introduces infrared renormalon ambiguities in matching coefficients and ultraviolet renormalon ambiguities in HQET matrix elements. When calculating matching coefficients $C,\ldots$, The integrals run over all loop momenta, including small ones. Therefore, they contain, in addition to the main short-distance contributions, also contributions from large distances, where the perturbation theory is ill-defined. They produce infrared renormalon singularities, factorially growing contributions to coefficients of the perturbative series, which lead to ambiguities $\sim (\Lambda_{\mathrm{QCD}}/m)^n$ in the matching coefficients $C,\ldots$ Similarly, HQET matrix elements of higher-dimensional operators $<O_i>,\ldots$ contain, in addition to the main large-distance contributions, also contributions from short distances, which produce ultraviolet-renormalon singularities. They lead to ambiguities of the order Λ^n_{QCD} times lower-dimensional matrix elements (e.g., $<\tilde{\jmath}>$). These two kinds of renormalon ambiguities should cancel in physical full QCD matrix elements $<j>$ (5.1) [46, 47].

Although this has been shown explicitly only in the large-β_0 limit, it is assumed to hold beyond this approximation. Based on this assumption, one may obtain additional information on the structure of the infrared renormalon singularities of matching coefficients, based on ultraviolet renormalons in higher-dimensional matrix elements, which are controlled by the renormalization group. This model-independent approach was applied to some simple HQET problems: the heavy-quark pole mass [100] and the chromomagnetic-interaction coefficient [96].

In this Chapter the heavy to light currents are investigated. The asymptotic behaviour of the perturbative series for the leading QCD/HQET matching coefficients (due to the nearest infrared renormalon) was studied in [46, 50, 51][2] in the large-β_0 limit. Here, an analysis of the heavy to light IR renormalons beyond the large-β_0 limit will be presented. In Sect. 5.1, an introduction of the IR renormalons in large-β_0 limit and how beyond it one can infer information of the asymptotic behaviour of the matching coefficients will be discussed. In Sect. 5.2, the cancellation of the IR renormalons against the UV renormalons of the subleading four dimension operator of the heavy to light currents will be shown explicitly by a direct calculation. Assuming that this cancellation holds beyond that limit, in Sect. 5.3 and Sect. 5.4 the asymptotic behaviour for the two Spin 0 currents and two Spin 1 currents will be given.

Ratios of meson matrix elements, such as f_{B^*}/f_B, are given by the ratios of the corresponding matching coefficients at the leading order in $1/m$. The asymptotic of the perturbative series for this ratio is studied in Sect. 5.5. The large two-loop correction in this ratio was observed in [50]; here a model-independent results for higher orders which continue this trend is presented.

---

[1] Otherwise, there may be several ultraviolet renormalons on the positive half-axis, leading to ambiguities of the order Λ^n_{QCD} times lower-dimensional matrix elements.

[2] Note a typo in (4.8) of [50]: denominators of both terms with a should be 2π, not π.

5.1 IR Renormalons

The leading order Wilson coefficients, C_Γ, of the heavy to light currents transitions can be expressed by a perturbative series in terms of the renormalization group invariant terms:

$$\hat{C}_\Gamma = 1 + \sum_{L=1}^{\infty} c_L^\Gamma \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L , \tag{5.2}$$

c_1^Γ and c_2^Γ are known. The asymptotic behaviour of this perturbative series c_L^Γ with $L \to \infty$ can be studied by analyzing the structure of the divergences in the Borel plain. The Borel transform is defined by,

$$S_\Gamma(u) = \sum_{L=1}^{\infty} \frac{c_L^\Gamma}{(L-1)!} \left(\frac{u}{\beta_0}\right)^{L-1} , \quad c_{n+1}^\Gamma = \left(\beta_0 \frac{d}{du}\right)^n S_\Gamma(u)\bigg|_{u=0} . \tag{5.3}$$

Formally, one can recover the expansion (5.2) by expanding around $u = 0$,

$$\hat{C}_\Gamma = 1 + \frac{1}{\beta_0} \int_0^\infty S_\Gamma(u) \, \exp\left(-\frac{4\pi}{\beta_0 \alpha_s(\mu_0)} u\right) du . \tag{5.4}$$

However, if $S_\Gamma(u)$ has singularities on the integration contour (which is the positive u axis), then the integral (5.4) is not well-defined, and the series (5.2) is not Borel-summable. To deal with these singularities some prescription is needed, leading to ambiguities in $\hat{C}_\Gamma$.

The matrix element of the full bare QCD current computed perturbatively can be expressed as:

$$\Gamma_0 = \Gamma \left[1 + \sum_{L=1}^{\infty} \sum_{n=0}^{L-1} a_{Ln} \beta_0^n \left(\frac{g_0^2}{(4\pi)^{d/2}}\right)^L\right] . \tag{5.5}$$

In the large β_0 limit where $\beta_0 \to \infty$ the leading contribution is given by $a_{L,L-1}$. This is determined by inserting $L-1$ quark loops into the gluon propagator in the one loop correction to the heavy to light matrix element. It can be shown that the perturbative matrix element can be written in terms of:

$$\Gamma_0 = \Gamma \left[1 + \frac{1}{\beta_0} \sum_{L=1}^{\infty} \frac{F(\epsilon, L\epsilon)}{L} \left(\frac{\beta}{\epsilon + \beta}\right)^L + \ldots\right] \tag{5.6}$$

where

$$F(\epsilon, u) = u e^{\gamma \epsilon} a_1(1 + u - \epsilon) \mu^{2u} D(\epsilon)^{u/\epsilon - 1}, \tag{5.7}$$

and $a_1(n)$ is the one loop expression of Γ_0 where the power of the gluon propagator has been risen to n, and $D(\epsilon) = 1 + (5/3)\epsilon$ one of the additional factors that come from the $L-1$ loop insertions. Renormalizing this perturbative series and matching with HQET where all loops vanish, if renormalize on-shell, the Wilson coefficients are given by the terms in the expansion with ϵ^0. From here, the perturbative series for the renormalization group invariant in the large-beta zero limit can be expressed as:

$$\hat{C}_\Gamma = 1 + \frac{1}{\beta_0} \int_0^\infty S_\Gamma(u) \, \exp\left(-\frac{4\pi}{\beta_0 \alpha_s(\mu_0)} u\right) du . \tag{5.8}$$

with,

$$S_\Gamma(\mu) = \left.\frac{F(0,u) - F(0,0)}{u}\right|_{\mu_0} . \tag{5.9}$$

$S_\Gamma(u)$ was calculated in [50],

$$S_\Gamma(u) = C_F \left\{ \frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)} \left[2(n-2)^2 - 2\eta(n-2)u + 3u^2 + u - 5\right] - \frac{2(n-2)^2 - 5}{2u} \right\} \tag{5.10}$$

(the results for the components of the vector current were obtained in [46]). Expanding this $S_\Gamma(u)$ in u reproduces leading large-β_0 terms in (5.2) (in particular, in c_1^Γ). This Borel image has IR renormalon poles at $u > 0$. The pole nearest to the origin (and thus giving the largest renormalon ambiguity) is situated at $u = \frac{1}{2}$,

$$S_\Gamma(u) = \frac{r_\Gamma}{\frac{1}{2} - u} + (\text{regular terms at } u = \tfrac{1}{2}) . \tag{5.11}$$

This leads to an ambiguity in the sum (5.4) of the series (5.2), the natural measure of which is the residue of the pole:

$$\Delta \hat{C}_\Gamma = \frac{r_\Gamma}{\beta_0} e^{5/6} \frac{\Lambda_{\overline{\rm MS}}}{m} ,$$

where $\Lambda_{\overline{\rm MS}}$ is for n_l flavours. This is commensurate with the $1/m$ corrections in (4.19). It is convenient to measure all such ambiguities in terms of the UV renormalon ambiguity of $\bar{\Lambda}$ [99]

$$\Delta \bar{\Lambda} = -2 C_F e^{5/6} \frac{\Lambda_{\overline{\rm MS}}}{\beta_0} . \tag{5.12}$$

Then [50],

$$\Delta \hat{C}_\Gamma = -\frac{1}{3} \left[2(n-2)^2 - \eta(n-2) - \frac{15}{4}\right] \frac{\Delta\bar{\Lambda}}{m} . \tag{5.13}$$

Beyond this limit, it will be shown that the simple pole becomes a branch point:

$$S_\Gamma(u) = \sum_i \frac{r_i}{\left(\frac{1}{2} - u\right)^{1+a_i}} + S_\Gamma^{reg}(u) \tag{5.14}$$

where a_i and r_i are unknown coefficient. $S_\Gamma^{reg}(u)$ is regular at $u = 1/2$. The IR renormalon ambiguity of $\hat{C}_\Gamma$ – generalizing the prescription to take the residue of the pole – is defined to be the integral of (5.14) around the cut divided by $2\pi i$:

$$\Delta \hat{C}_\Gamma = \frac{1}{\beta_0} \exp\left[-\frac{2\pi}{\beta_0 \alpha_s(\mu_0)}\right] \sum_i \frac{r_i}{\Gamma(1+a_i)} \left(\frac{\beta_0 \alpha_s(\mu_0)}{4\pi}\right)^{-a_i} . \tag{5.15}$$

Requiring the cancellation of renormalons ambiguities, one obtains a_i and r_i in terms of known perturbative coefficients. Therefore, expanding (5.14) as (5.3), one can obtain the asymptotic behaviour of the perturbative series

$$c_{n+1}^\Gamma = r_i 2^{1+a_i} (2\beta_0)^n \Gamma(1+a_i)^{-1} n! n^{a_i} (1 + \mathcal{O}(1/n)) \tag{5.16}$$

$S_\Gamma^{reg}(u)$ has singularities at $u = 1$ (IR) and $u = -1$(UV), and thus gives exponentially smaller contributions with $(\pm\beta_0)^n$ instead of $(2\beta_0)^n$.

5.2 UV Renormalons

In this section, the UV renormalons for the dimension four operators in the large β_0 limit will be calculated for the first time by a direct calculation, checking the cancellation of the UV renormalon against the IR ones of the leading order Wilson coefficients.

Finally, the UV renormalon ambiguities will be generalized beyond the large-β_0 limit.

Ultraviolet contributions to the matrix elements of $O_{3,4}$ are independent of the external states, and one may use quarks instead of hadron states (see (4.69)). By dimensional analysis, the UV renormalon ambiguities of the matrix elements of $O_{3,4}$ are proportional to $\Delta\bar{\Lambda}$ times the matrix element of the lower-dimensional operator $\tilde{\jmath}$ with the same external states. In order to avoid IR divergences, it is enough to consider transitions from an off-shell heavy quark with residual energy $\omega < 0$ to a light quark with zero momentum. For O_3, all loop corrections to the vertex function (see Fig. 5.1) vanish. The kinetic-energy vertices contain no Dirac matrices, and one can take $\frac{1}{4}$ of the trace on the light-quark line; this yields k^α at the vertex, and the gluon propagator with insertions is transverse. There is one more contribution [96] which have to be taken into account. The matrix element F of $\tilde{\jmath}$ should be multiplied by the heavy-quark wave-function renormalization $Z_h^{1/2}$, which contains a kinetic-energy contribution. This contribution is known to have an UV renormalon ambiguity [96]

$$\Delta Z_h = -\frac{3}{2}\frac{\Delta\bar{\Lambda}}{m},$$

thus giving $-(3/4)(\Delta\bar{\Lambda}/m)F$ as the ambiguity of the matrix element of O_3. This must be equal to $F \cdot \Delta G_k/(2m)$. Therefore:

$$\Delta G_k(\mu) = -\frac{3}{2}\Delta\bar{\Lambda}, \tag{5.17}$$

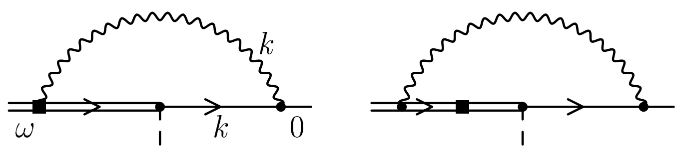

Figure 5.1: Matrix element of O_3; renormalon chains are inserted into the gluon propagators

For O_4, a straightforward calculation of the diagram similar to the first one in Fig. 5.1 gives the bare matrix element of the usual form (see [50, 96])

$$\Gamma\left[1 + \frac{1}{\beta_0}\sum_{L=1}^{\infty}\frac{F(\varepsilon, L\varepsilon)}{L}\left(\frac{\beta}{\varepsilon+\beta}\right)^L + \mathcal{O}\left(\frac{1}{\beta_0^2}\right)\right]$$

with $\beta = \beta_0\alpha_s(\mu)/(4\pi)$,

$$F(\varepsilon, u) = -2d_\Gamma C_F C_m^{\text{bare}}\frac{\omega}{m}\left(\frac{\mu}{-2\omega}\right)^{2u} e^{\gamma\varepsilon}\frac{u\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(2+u-\varepsilon)}D(\varepsilon)^{\frac{u}{\varepsilon}-1},$$

where

$$D(\varepsilon) = 6e^{\gamma\varepsilon}\Gamma(1+\varepsilon)B(2-\varepsilon,2-\varepsilon) = 1 + \frac{5}{3}\varepsilon + \cdots\,,$$
$$\sigma_{\mu\nu}\Gamma\frac{1+\not{v}}{2}\sigma^{\mu\nu}\frac{1+\not{v}}{2} = 2d_\Gamma\Gamma\frac{1+\not{v}}{2}\,,$$
$$d_\Gamma = \frac{1}{2}\left[(1-2\eta(n-2))^2 - 3\right]\,.$$

The renormalization-group invariant matrix element has the form (5.4) with $\mu_0 = -2\omega e^{-5/6}$ and

$$S(u) = \left.\frac{F(0,u)-F(0,0)}{u}\right|_{\mu=\mu_0}$$
$$= -2d_\Gamma C_F C_m(-2\omega)\frac{\omega}{m}\left(\frac{\Gamma(-1+2u)\Gamma(1-u)}{\Gamma(2+u)} + \frac{1}{2u}\right)\,.$$

Taking the residue at the pole $u = \frac{1}{2}$, one obtains the UV renormalon ambiguity $(d_\Gamma/3)C_m(-2\omega)(\Delta\bar\Lambda/m)$ times the matrix element of $\tilde\jmath$. For B-meson, $d_\Gamma = 3$; comparing with (4.73) gives:

$$\Delta G_m(\mu) = 2\Delta\bar\Lambda \tag{5.18}$$

From (4.72), (5.17) and (5.18), one obtains:

$$\Delta\hat G_k = -\frac{3}{2}\Delta\bar\Lambda\,,\quad \Delta\hat G_m = \left(2 - \frac{\gamma_0^m}{\gamma_{m0}}\right)\Delta\bar\Lambda\,. \tag{5.19}$$

In the $1/m$ expansion for the meson decay constants (4.75), $\Gamma = 1$ and $\not{v}$ correspond to $n = 0$, $\eta = -1$ and $n = 1$, $\eta = 1$ in (5.13), and the IR renormalon ambiguities (5.13) are $-(3/4)(\Delta\bar\Lambda/m)$ and $(1/4)(\Delta\bar\Lambda/m)$. They cancel with the UV renormalon ambiguities of the hadronic matrix elements (5.12), (5.19) in the $1/m$ corrections in (4.75). In fact, the results (5.17) and (5.18) were first obtained [46] from the *requirement* of such cancellation for $\Gamma = \not{v}$, $\gamma_\perp$, by solving a system of two linear equations, and later confirmed [50] by considering all possible Γ (this gives three equations, thus providing a consistency check). Here the cancellation of renormalon ambiguities by a direct calculation has been shown.

This cancellation should hold also beyond the large β_0 limit. By dimensional arguments, the UV renormalon ambiguities of the μ-independent hadronic parameters $\bar\Lambda$, $\hat G_k$, $\hat G_m$ must be equal to $\Lambda_{\overline{\rm MS}}$ times some numbers:

$$\Delta\bar\Lambda = N_0\Delta_0\,,\quad \Delta\hat G_k = -\frac{3}{2}N_1\Delta_0\,,\quad \Delta\hat G_m = N_2\left(2 - \frac{\gamma_0^m}{\gamma_{m0}}\right)\Delta_0\,,$$
$$\Delta_0 = -2C_F e^{5/6}\frac{\Lambda_{\overline{\rm MS}}}{\beta_0}\,. \tag{5.20}$$

The normalization factors are unity in the large β_0 limit, $N_i = 1 + \mathcal{O}(1/\beta_0)$; in general, they are just some unknown numbers of order one. Using

$$\Lambda_{\overline{\rm MS}} = \mu_0 \exp\left[-\frac{2\pi}{\beta_0\alpha_s(\mu_0)}\right]\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{-\frac{\beta_1}{2\beta_0^2}}$$
$$\times\left[1 - \frac{\beta_0\beta_2 - \beta_1^2}{2\beta_0^3}\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right]\,,\quad \mu_0 = e^{-5/6}m \tag{5.21}$$

(see (2.26)), One can represent the UV renormalon ambiguities of the $1/m$ corrections as $\exp[-2\pi/(\beta_0\alpha_s(\mu_0))]$ times a sum of terms with different fractional powers of $\alpha_s(\mu_0)/(4\pi)$.[3] In order to cancel this ambiguity, one should have the branching point

$$S_\Gamma(u) = \sum_i \frac{r_i}{\left(\frac{1}{2}-u\right)^{1+a_i}} \tag{5.22}$$

instead of a simple pole (5.11).

In the next Sections, base on this assumptions the asymptotic behaviour for the leading matching coefficients of heavy to light current transitions will be presented.

5.3 Spin 0

At the next-to-leading order, recovering the result (4.75) of previous chapter, one has:

$$\begin{aligned}\left\{\begin{array}{c}\hat{f}_B^P\\ f_B\end{array}\right\} = \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\tilde{\gamma}_0}{2\beta_0}} \frac{\hat{C}_\Gamma \hat{F}}{\sqrt{m_B}}\Bigg\{1 \\ + \frac{1}{2m}\Bigg[\left(-\frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu_0)}{4\pi} \pm 1 + \frac{\gamma_0^m}{\gamma_{m0}} + c_{\Lambda 1}^\Gamma \frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right)\bar{\Lambda} \\ + \hat{G}_k + \hat{G}_m\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left(1 + c_{m1}\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right)\Bigg]\Bigg\},\end{aligned} \tag{5.23}$$

where

$$\begin{aligned}c_{\Lambda 1}^\Gamma = (1\mp 1)\left(c_1^{\not{v}} - c_1^{\gamma_\perp}\right) - b_{21}^\Gamma + \frac{\gamma_0^m}{\gamma_{m0}}c_{m1} - \frac{\gamma_1^k+\gamma_1^m}{2\beta_0} + \frac{\beta_1\left(\gamma_0^k+\gamma_0^m\right)}{2\beta_0^2} \\ + \frac{\gamma_{m0}\gamma_1^m - \gamma_0^m\gamma_{m1}}{2\beta_0\left(\gamma_{m0}-2\beta_0\right)}\end{aligned}$$

(here $\gamma_0^k+\gamma_0^m = 0$). The left part of the equation has no renormalon ambiguities. Therefore, assuming the cancellation of the IR renormalons ambiguities of the Wilson coefficients (5.15) against the UV ones (5.20) one obtains an equality from which the unknown coefficient a_i and r_i can be extracted

$$\Delta\hat{C}_\Gamma\{\ldots\} + \hat{C}_\Gamma[\ldots]\Delta_0 = 0 \tag{5.24}$$

Once a_i and r_i are known, one can obtain the Borel transform in terms of them (5.14):

$$\begin{aligned}S_\Gamma(u) = \frac{C_F}{\left(\frac{1}{2}-u\right)^{1+\frac{\beta_1}{2\beta_0^2}}}\left[1 + c_1^{\Gamma\prime}\left(\tfrac{1}{2}-u\right)+\cdots\right]\Bigg\{ \\ \left[-\frac{\gamma_0^k}{2\beta_0}\left(\log\frac{\frac{1}{2}-u}{\beta_0} - \psi\left(1+\frac{\beta_1}{2\beta_0^2}\right)\right) \pm 1 + \frac{\gamma_0^m}{\gamma_{m0}} + c_{\Lambda 1}^{\Gamma\prime}\left(\tfrac{1}{2}-u\right)+\cdots\right]N_0' \\ -\frac{3}{2}N_1' + \left(2-\frac{\gamma_0^m}{\gamma_{m0}}\right)N_2'\left(\tfrac{1}{2}-u\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left[1 + c_{m1}'\left(\tfrac{1}{2}-u\right)+\cdots\right]\Bigg\},\end{aligned} \tag{5.25}$$

[3]It is convenient to replace $\log[\alpha_s(\mu_0)/(4\pi)] \to [(\alpha_s(\mu_0)/(4\pi))^\delta - 1]/\delta$, and take the limit $\delta \to 0$ at the end of calculation.

where

$$\frac{N_0'}{N_0} = \frac{N_1'}{N_1} = \Gamma\left(1 + \frac{\beta_1}{2\beta_0^2}\right) \beta_0^{\frac{\beta_1}{2\beta_0^2}}, \quad \frac{N_2'}{N_2} = \Gamma\left(1 + \frac{\beta_1}{2\beta_0^2} - \frac{\gamma_{m0}}{2\beta_0}\right) \beta_0^{\frac{\beta_1}{2\beta_0^2} - \frac{\gamma_{m0}}{2\beta_0}},$$
$$c_1^{\Gamma\prime} = \frac{2\beta_0}{\beta_1}\left(c_1^\Gamma - \frac{\beta_0\beta_2 - \beta_1^2}{2\beta_0^3}\right), \quad c_{\Lambda 1}^{\Gamma\prime} = \frac{2\beta_0}{\beta_1}\left(c_{\Lambda 1}^\Gamma - \tfrac{1}{2}\gamma_0^k c_1^{\Gamma\prime}\right),$$
$$c_{m1}' = \frac{\beta_0\left(2c_{m1} + \gamma_{m0} c_1^{\Gamma\prime}\right)}{\beta_1 - \beta_0\gamma_{m0}}. \tag{5.26}$$

In the large β_0 limit, the formula (5.25) reproduces the known results (5.11), (5.13).

The asymptotic of c_L^Γ (4.12) at $L \gg 1$ is determined by the renormalon singularity closest to the origin (see (5.3)). At $n \gg 1$,

$$\Gamma(n + a + 1) = n!\, n^a \left(1 + \frac{a(a+1)}{2n} + \cdots\right),$$

and one arrives at

$$\begin{aligned} c_{n+1}^\Gamma = 2C_F\, n!\, (2\beta_0)^n\, (2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}} \left(1 + \frac{c_1^{\Gamma\prime\prime}}{2\beta_0 n} + \cdots\right) \Bigg[\\ \left(\frac{\gamma_0^k}{2\beta_0}\log 2\beta_0 n \pm 1 + \frac{\gamma_0^m}{\gamma_{m0}} + \frac{c_{\Lambda 1}^{\Gamma\prime\prime}}{2\beta_0 n} + \cdots\right) N_0 \\ - \frac{3}{2}N_1 + \left(2 - \frac{\gamma_0^m}{\gamma_{m0}}\right) N_2 (2\beta_0 n)^{-\frac{\gamma_{m0}}{2\beta_0}} \left(1 + \frac{c_{m1}''}{2\beta_0 n} + \cdots\right)\Bigg], \end{aligned} \tag{5.27}$$

where

$$c_1^{\Gamma\prime\prime} = c_1^\Gamma - \frac{2\beta_0\beta_2 - 2\beta_0^2\beta_1 - 3\beta_1^2}{4\beta_0^3}, \quad c_{\Lambda 1}^{\Gamma\prime\prime} = c_{\Lambda 1}^\Gamma + \gamma_0^k \frac{\beta_1 + \beta_0^2}{2\beta_0^2},$$
$$c_{m1}'' = c_{m1} - \gamma_{m0}\frac{2(\beta_1 + \beta_0^2) - \beta_0\gamma_{m0}}{4\beta_0^2}. \tag{5.28}$$

The result (5.27) is model-independent and the powers of n are exact. However, the normalization factors N_i cannot be determined within this approach. The leading term at $n \to \infty$ formally is the logarithmic term, because γ_{m0} is positive (see (2.22)). At moderate values of n, all leading terms are of similar importance.

The ratio of the decay constants taken from (4.77) is:

$$\frac{\hat{f}_B^P}{f_B} = \frac{\hat{C}_1}{\hat{C}_{\not{v}}}\left(1 + \frac{\bar{\Lambda}}{m}\right). \tag{5.29}$$

where $\hat{C}_1/\hat{C}_{\not{v}}$ is the ratio of the quark masses given by (4.60). Naturally, it contains no $1/m$ corrections with B-meson hadronic parameters; it is just a series in $\alpha_s(\mu_0)$, see Appendix A. The Borel image $S(u)$ of this series is

$$S(u) = \frac{2C_F N_0'}{\left(\frac{1}{2} - u\right)^{1 + \frac{\beta_1}{2\beta_0^2}}}\left[1 + c_1^{(m/\hat{m})'}\left(\tfrac{1}{2} - u\right) + c_2^{(m/\hat{m})'}\left(\tfrac{1}{2} - u\right)^2 \cdots\right],$$

$$
\begin{aligned}
c_1^{(m/\hat{m})'} &= \frac{2\beta_0}{\beta_1}\left(c_1^{(m/\hat{m})} - \frac{\beta_0\beta_2 - \beta_1^2}{2\beta_0^3}\right), \\
c_2^{(m/\hat{m})'} &= \frac{4\beta_0^2}{\beta_1(\beta_1 - 2\beta_0^2)} \\
&\quad \times \left[c_2^{(m/\hat{m})} - \frac{\beta_0\beta_2 - \beta_1^2}{2\beta_0^3} c_1^{(m/\hat{m})} \right. \\
&\quad \left. - \frac{2\beta_0^4\beta_3 - \beta_0^2\beta_2^2 + \beta_1(\beta_1 - 2\beta_0^2)(2\beta_0\beta_2 - \beta_1^2)}{8\beta_0^6}\right],
\end{aligned}
\tag{5.30}
$$

and the asymptotics of $c_L^{(m/\hat{m})}$ at $L \gg 1$ is

$$
c_{n+1}^{(m/\hat{m})} = 4C_F N_0\, n!\,(2\beta_0)^n\,(2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}} \left(1 + \frac{c_1^{(m/\hat{m})''}}{2\beta_0 n} + \frac{c_2^{(m/\hat{m})''}}{(2\beta_0 n)^2} + \cdots\right), \tag{5.31}
$$

$$
\begin{aligned}
c_1^{(m/\hat{m})''} &= c_1^{(m/\hat{m})} - \frac{2\beta_0\beta_2 - 2\beta_0^2\beta_1 - 3\beta_1^2}{4\beta_0^3} \\
c_2^{(m/\hat{m})''} &= c_2^{(m/\hat{m})} - \frac{2\beta_0\beta_2 + 2\beta_0^2\beta_1 - \beta_1^2}{4\beta_0^3} c_1^{(m/\hat{m})} \\
&\quad - \frac{24\beta_0^4\beta_3 - 12\beta_0^2\beta_2^2 + \beta_1(\beta_1 - 2\beta_0^2)(36\beta_0\beta_2 - 27\beta_1^2 - 10\beta_0^2\beta_1 - 8\beta_0^4)}{96\beta_0^6}.
\end{aligned}
$$

These results are equivalent to [100].

It is possible to estimate the size of N_0 following [128–130]. The function

$$
\begin{aligned}
\tilde{S}_{m/\hat{m}}(u) &= (1-2u)^{1+\frac{\beta_1}{2\beta_0^2}} S_{m/\hat{m}}(u), \\
S_{m/\hat{m}}(u) &= \sum_{L=1}^{\infty} \frac{c_L^{(m\hat{m})}}{(L-1)!}\left(\frac{u}{\beta_0}\right)^{L-1}
\end{aligned}
\tag{5.32}
$$

still has a singularity at $u = 1/2$ due to $S_\Gamma^{reg}(u)$ in (5.14), but has a finite limit at $u \to 1/2 - 0$. The radius of convergence of its expansion in u is $1/2$, but the series should converge at $u = 1/2$. Therefore, one can calculate $\tilde{S}_{m/\hat{m}}(1/2)$, and hence N_0, from this expansion. Substituting $c_L^{(m/\hat{m})}$ for $L \gg 3$ from Appendix A, one finds

$$
N_0 \sim 0.288 \cdot (1 + 0.075 + 0.630 + \ldots) = 0.491 \tag{5.33}
$$

The three loop correction turns out to be large, which casts some doubts on this estimate of N_0. Next, the matrix elements with Spin 1 will be studied.

5.4 Spin 1

At the next-to-leading order from (4.110) :

$$
\left\{\begin{array}{c} f_{B^*} \\ \hat{f}_{B^*}^T \end{array}\right\} = \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\tilde{\gamma}_0}{2\beta_0}} \frac{\hat{C}_\Gamma \hat{F}}{\sqrt{m_{B^*}}} \Bigg\{ 1
$$

$$+\frac{1}{2m}\left[\left(-\frac{\gamma_0^k}{2\beta_0}\log\frac{\alpha_s(\mu_0)}{4\pi}+\frac{1}{3}\left(\pm 1-\frac{\gamma_0^m}{\gamma_{m0}}\right)+c_{\Lambda 1}^\Gamma\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\right)\bar\Lambda\right.$$
$$\left.\left.+\hat G_k-\frac{1}{3}\hat G_m\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left(1+c_{m1}\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\right)\right]\right\}, \qquad (5.34)$$

where

$$c_{\Lambda 1}^\Gamma=\tfrac{1}{3}(1\pm 1)\left(c_1^{\gamma_\perp}-c_1^{\not v}\right)-\tfrac{1}{3}b_{31}^\Gamma-b_{41}^\Gamma-\tfrac{1}{3}\frac{\gamma_0^m}{\gamma_{m0}}c_{m1}\mp\tfrac{2}{9}\gamma_{a0}+\tfrac{1}{3}\gamma_{10}^m+\tfrac{4}{3}\gamma_{20}^m$$
$$-\frac{\gamma_1^k-\frac{1}{3}\gamma_1^m}{2\beta_0}+\frac{\beta_1\left(\gamma_0^k-\frac{1}{3}\gamma_0^m\right)}{2\beta_0^2}-\frac{\gamma_{m0}\gamma_1^m-\gamma_0^m\gamma_{m1}}{6\beta_0\left(\gamma_{m0}-2\beta_0\right)}.$$

In the large-β_0 limit, the IR renormalon ambiguities $\Delta\hat C_\Gamma$ (5.13) for $\Gamma=\gamma_\perp$, $\gamma_\perp\not v$ (having $n=1$, $\eta=-1$ and $n=2$, $\eta=1$) are $(11/12)(\Delta\bar\Lambda/m)$ and $(5/4)(\Delta\bar\Lambda/m)$. They cancel with the UV renormalon ambiguities of the hadronic matrix elements (5.12), (5.19) in the $1/m$ corrections in (5.34).

Beyond the large β_0-limit, requiring the cancellation of the renormalon ambiguities and extracting the unknown parameters, the Borel images are

$$S_\Gamma(u)=\frac{C_F}{\left(\frac{1}{2}-u\right)^{1+\frac{\beta_1}{2\beta_0^2}}}\left[1+c_1^{\Gamma\prime}\left(\tfrac{1}{2}-u\right)+\cdots\right]\left\{\right.$$
$$\left[-\frac{\gamma_0^k}{2\beta_0}\left(\log\frac{\frac{1}{2}-u}{\beta_0}-\psi\left(1+\frac{\beta_1}{2\beta_0^2}\right)\right)\right.$$
$$\left.+\frac{1}{3}\left(\pm 1-\frac{\gamma_0^m}{\gamma_{m0}}\right)+c_{\Lambda 1}^{\Gamma\prime}\left(\tfrac{1}{2}-u\right)+\cdots\right]N_0'$$
$$\left.-\frac{3}{2}N_1'-\frac{1}{3}\left(2-\frac{\gamma_0^m}{\gamma_{m0}}\right)N_2'\left(\tfrac{1}{2}-u\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left[1+c_{m1}'\left(\tfrac{1}{2}-u\right)+\cdots\right]\right\}, \qquad (5.35)$$

(see (5.26)), and the perturbative coefficients at $n\gg 1$ are

$$c_{n+1}^\Gamma=2C_F\,n!\,(2\beta_0)^n\,(2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}}\left(1+\frac{c_1^{\Gamma\prime\prime}}{2\beta_0 n}+\cdots\right)\left[\right.$$
$$\left(\frac{\gamma_0^k}{2\beta_0}\log 2\beta_0 n+\frac{1}{3}\left(\pm 1-\frac{\gamma_0^m}{\gamma_{m0}}\right)+\frac{c_{\Lambda 1}^{\Gamma\prime\prime}}{2\beta_0 n}+\cdots\right)N_0$$
$$\left.-\frac{3}{2}N_1-\frac{1}{3}\left(2-\frac{\gamma_0^m}{\gamma_{m0}}\right)N_2(2\beta_0 n)^{-\frac{\gamma_{m0}}{2\beta_0}}\left(1+\frac{c_{m1}''}{2\beta_0 n}+\cdots\right)\right] \qquad (5.36)$$

(see (5.28)).

The ratio $\hat f_{B^*}^T/f_{B^*}$ at the leading order in $1/m$ is given by the perturbative series in $\alpha_s(\mu_0)$ with

$$c_1=C_F\left[-\tfrac{1}{2}+\left(\tfrac{25}{2}C_F-\tfrac{4}{3}C_A\right)\frac{1}{\beta_0'}-(11C_F+7C_A)\frac{C_A}{\beta_0'^2}\right], \qquad (5.37)$$
$$c_2=C_F\left\{\left(-\tfrac{1}{9}\pi^2+\tfrac{89}{54}\right)\beta_0'+\left(2\zeta_3+\tfrac{8}{3}\pi^2\log 2-\tfrac{35}{9}\pi^2+\tfrac{139}{24}\right)C_F\right.$$

$$+\left(-4\zeta_3-\tfrac{4}{3}\pi^2\log 2+\tfrac{47}{27}\pi^2-\tfrac{19955}{1296}\right)C_A+\left(-\tfrac{28}{27}\pi^2+\tfrac{619}{81}\right)T_F$$
$$+\Big[\left(32\zeta_3-\tfrac{337}{6}\right)C_F^2+\left(-78\zeta_3+\tfrac{151}{4}\right)C_FC_A+\left(42\zeta_3-\tfrac{131}{36}\right)C_A^2$$
$$-\tfrac{250}{9}C_FT_F+\tfrac{80}{27}C_AT_F\Big]\frac{1}{\beta_0'}$$
$$+\Big(\tfrac{625}{8}C_F^3+\tfrac{461}{6}C_F^2C_A+\tfrac{2375}{36}C_FC_A^2-\tfrac{511}{48}C_A^3$$
$$+\tfrac{220}{9}C_FC_AT_F+\tfrac{140}{9}C_A^2T_F\Big)\frac{1}{\beta_0'^2}$$
$$-\tfrac{1}{6}\left(75C_F^2+25C_FC_A+21C_A^2\right)(11C_F+7C_A)\frac{C_A}{\beta_0'^3}$$
$$+\tfrac{1}{2}C_F\left(11C_F+7C_A\right)^2\frac{C_A^2}{\beta_0'^4}\Big\}$$

(from the result in [50], omitting the $m_c \neq 0$ effect, and the three-loop anomalous dimension γ'_σ of the tensor current [114]). This ratio is, from (4.106),

$$\frac{\hat{f}^T_{B^*}}{f_{B^*}}=\frac{\hat{C}_{\gamma_\perp \not{v}}}{\hat{C}_{\gamma_\perp}}\left[1-\frac{\bar{\Lambda}}{3m}\left(1+c_{\Lambda 1}\frac{\alpha_s(\mu_0)}{4\pi}+\cdots\right)\right], \tag{5.38}$$
$$c_{\Lambda 1}=\tfrac{3}{2}\left(c_{\Lambda 1}^{\gamma_\perp\not{v}}-c_{\Lambda 1}^{\gamma_\perp}\right)=c_1^{\gamma_\perp}-c_1^{\not{v}}+\tfrac{1}{2}\left(b_{31}^{\gamma_\perp\not{v}}-b_{31}^{\gamma_\perp}\right)+\tfrac{3}{2}\left(b_{41}^{\gamma_\perp\not{v}}-b_{41}^{\gamma_\perp}\right)-\tfrac{2}{3}\gamma_{a0}\,.$$

Therefore, the Borel image of the perturbative series is

$$S(u)=-\frac{2}{3}\frac{C_FN_0'}{\left(\frac{1}{2}-u\right)^{1+\frac{\beta_1}{2\beta_0^2}}}\left[1+c_1'\left(\tfrac{1}{2}-u\right)+\cdots\right], \tag{5.39}$$
$$c_1'=\frac{2\beta_0}{\beta_1}\left(c_{\Lambda 1}+c_1-\frac{\beta_0\beta_2-\beta_1^2}{2\beta_0^3}\right),$$

and the asymptotics of c_L at $L \gg 1$ is

$$c_{n+1}=-\frac{4}{3}C_FN_0\,n!\,(2\beta_0)^n\,(2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}}\left(1+\frac{c_1''}{2\beta_0 n}+\cdots\right), \tag{5.40}$$
$$c_1''=c_{\Lambda 1}+c_1-\frac{2\beta_0\beta_2-2\beta_0^2\beta_1-3\beta_1^2}{4\beta_0^3}\,.$$

5.5 Results and Conclusion

The main result is for the ratio of two (in principle) measurable quantities, f_{B^*}/f_B. It is given by the perturbative series in $\alpha_s(\mu_0)$ with

$$c_1=-2C_F\,, \tag{5.41}$$
$$c_2=C_F\Big[-3\beta_0+\left(-8\zeta_3+\tfrac{16}{3}\pi^2\log 2-\tfrac{64}{9}\pi^2+\tfrac{31}{3}\right)C_F$$
$$+\left(4\zeta_3-\tfrac{8}{3}\pi^2\log 2+\tfrac{8}{3}\pi^2-6\right)C_A+\left(\tfrac{16}{9}\pi^2-\tfrac{164}{9}\right)T_F\Big]\,.$$

(from [50], omitting the $m_c \neq 0$ effect). This ratio is, from (5.23) and (5.34),

$$\frac{f_{B^*}}{f_B} = \frac{\hat{C}_{\gamma_\perp}}{\hat{C}_{\not{v}}}\Bigg\{1 + \frac{2}{3m}\Bigg[\left(1 - \frac{\gamma_0^m}{\gamma_{m0}} + c_{\Lambda 1}\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right)\bar{\Lambda} - \hat{G}_m\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left(1 + c_{m1}\frac{\alpha_s(\mu_0)}{4\pi} + \cdots\right)\Bigg]\Bigg\}, \tag{5.42}$$

where

$$c_{\Lambda 1} = \frac{3}{4}\left(c_{\Lambda 1}^{\gamma_\perp} - c_{\Lambda 1}^{\not{v}}\right) = 6C_F\left[-1 - \frac{4}{3}\frac{C_F}{C_A} + \frac{8(4\pi^2+3)C_F - (8\pi^2 - 93)C_A}{27(\beta_0 - C_A)}\right].$$

The Borel image of the perturbative series is

$$S(u) = \frac{4}{3}\frac{C_F}{\left(\frac{1}{2} - u\right)^{1+\frac{\beta_1}{2\beta_0^2}}}\Bigg[\left(1 - \frac{\gamma_0^m}{\gamma_{m0}} + c'_{\Lambda 1}\left(\tfrac{1}{2} - u\right) + \cdots\right)N'_0 - \left(2 - \frac{\gamma_0^m}{\gamma_{m0}}\right)N'_2\left(\tfrac{1}{2} - u\right)^{\frac{\gamma_{m0}}{2\beta_0}}\left(1 + c'_{m1}\left(\tfrac{1}{2} - u\right) + \cdots\right)\Bigg], \tag{5.43}$$

where

$$c'_{\Lambda 1} = \frac{2\beta_0}{\beta_1}\left[c_{\Lambda 1} + \left(1 - \frac{\gamma_0^m}{\gamma_{m0}}\right)c'_1\right], \quad c'_{m1} = \frac{2\beta_0}{\beta_1 - \beta_0\gamma_{m0}}\left(c_{m1} + c'_1\right),$$
$$c'_1 = c_1 - \frac{\beta_0\beta_2 - \beta_1^2}{2\beta_0^3}.$$

The asymptotics of the coefficients is

$$c_{n+1} = \frac{8}{3}C_F\, n!\,(2\beta_0)^n(2\beta_0 n)^{\frac{\beta_1}{2\beta_0^2}}\Bigg[\left(1 - \frac{\gamma_0^m}{\gamma_{m0}} + \frac{c''_{\Lambda 1}}{2\beta_0 n} + \cdots\right)N_0 - \left(2 - \frac{\gamma_0^m}{\gamma_{m0}}\right)N_2(2\beta_0 n)^{-\frac{\gamma_{m0}}{2\beta_0}}\left(1 + \frac{c''_{m1}}{2\beta_0 n} + \cdots\right)\Bigg], \tag{5.44}$$

where

$$c''_{\Lambda 1} = c_{\Lambda 1} + \left(1 - \frac{\gamma_0^m}{\gamma_{m0}}\right)\left(c'_1 + \frac{\beta_1(\beta_1 + 2\beta_0^2)}{4\beta_0^3}\right),$$
$$c''_{m1} = c_{m1} + c'_1 + \frac{(\beta_1 - \beta_0\gamma_{m0})(\beta_1 + 2\beta_0^2 - \beta_0\gamma_{m0})}{4\beta_0^3}.$$

Substituting numerical values, the perturbative series is

$$\frac{f_{B^*}}{f_B} = 1 - \frac{2}{3}\frac{\alpha_s(\mu_0)}{\pi} - \left(-\frac{1}{9}\zeta_3 + \frac{2}{27}\pi^2\log 2 + \frac{4}{81}\pi^2 + \frac{115}{36}\right)\left(\frac{\alpha_s(\mu_0)}{\pi}\right)^2 + \cdots + \mathcal{O}\left(\frac{\Lambda_{\overline{\rm MS}}}{m_b}\right). \tag{5.45}$$

The asymptotics (5.44) becomes

$$c_{n+1} = -\frac{224}{81} n! \left(\frac{50}{3}\right)^n \left(\frac{50}{3} n\right)^{\frac{231}{625}} \left\{ \left[1 - \frac{2}{25}\left(\pi^2 + \frac{924493}{250000}\right)\frac{1}{n} + \cdots\right] N_0 \right.$$
$$\left. + \frac{2}{7} N_2 \left(\frac{50}{3} n\right)^{-\frac{9}{25}} \left(1 + \frac{40157}{3125000}\frac{1}{n} + \cdots\right)\right\}. \quad (5.46)$$

Definite quantitative predictions cannot be made, because the normalization coefficients $N_{0,2}$ are unknown. Table 5.1 shows the growth of the coefficients . The coefficients $c_L/4^L$ of $\alpha_s(\mu_0)/\pi$ are given in three columns. The first column shows the exactly known ones (5.45). The second column shows the results of the large-β_0 limit. The Borel image $S(u)$ in this limit is, from (5.10),

$$S(u) = -4C_F \frac{\Gamma(1+u)\Gamma(1-2u)}{\Gamma(3-u)}. \quad (5.47)$$

Expanding it at $u = 0$ (5.3), one gets:

$$c_1 = -2C_F, \quad c_2 = -3C_F\beta_0, \quad c_3 = -C_F\left(\tfrac{4}{3}\pi^2 + 7\right)\beta_0^2,$$
$$c_4 = -3C_F\left(8\zeta_3 + 2\pi^2 + \tfrac{15}{2}\right)\beta_0^3, \quad c_5 = -C_F\left(144\zeta_3 + \tfrac{24}{5}\pi^4 + 28\pi^2 + 93\right)\beta_0^4,$$
$$c_6 = -C_F\left(1440\zeta_5 + 160\pi^2\zeta_3 + 840\zeta_3 + 36\pi^4 + 150\pi^2 + \tfrac{945}{2}\right)\beta_0^5, \ldots$$

This limit reproduces c_1 and the β_0-term of c_2 (5.41). Finally, the third column shows the asymptotics (5.46) at $N_0 = N_2 = 1$. Let's stress once more that this is *not* the result of QCD, but simply a numerical illustration of the typical behaviour of the perturbative coefficients. The large-β_0 result includes not just the pole at $u = \frac{1}{2}$, but the whole function $S(u)$ (5.47). In contrast to this, the asymptotics (5.46) is determined by the nearest singularity at $u = \frac{1}{2}$ only, but includes all powers of β_0, not just the highest one. To show the rate of convergence of the $1/n$ expansion (5.46), the next column shows the ratio of the sum of the subleading (i.e., $1/n$ suppressed) terms to the sum of the leading ones. One can see that the reliability of our next-to-leading order results at $L \lesssim 10$ is not high. Finally, the last column shows the complete L-loop contribution to f_{B^*}/f_B, according to (5.46) with $N_0 = N_2 = 1$. The value $\alpha_s(e^{-5/6}m_b) = 0.299$ has been obtained using RunDec [131]. The smallest contribution seems to be the 3-loop one, and it is about 4% (though this small value is due to a partial cancellation between the leading order and the next-to-leading one, and thus is not quite reliable). Therefore, calculation of this 3-loop correction is meaningful (and it is actually possible, using the technique of [132, 133]), while there would be no sense in the 4-loop calculation.

For the ratio $m/\hat{m}$, the result, using Appendix A, is:

$$\frac{m}{\hat{m}} = 1 + \tfrac{891}{1058}\frac{\alpha_s(\mu_0)}{\pi} + \left(-\tfrac{173}{138}\zeta_3 + \tfrac{1}{9}\pi^2\log 2 + \tfrac{1}{9}\pi^2 + \tfrac{168550145}{40297104}\right)\left(\frac{\alpha_s(\mu_0)}{\pi}\right)^2$$
$$+ \left(-\tfrac{188}{27}a_4 + \tfrac{50225}{4968}\zeta_5 - \tfrac{1439}{432}\pi^2\zeta_3 - \tfrac{4396763}{438012}\zeta_3 - \tfrac{47}{162}\log^4 2 - \tfrac{14}{81}\pi^2\log^2 2\right.$$
$$\left. - \tfrac{402485}{85698}\pi^2\log 2 + \tfrac{461}{7776}\pi^4 + \tfrac{220317449}{20567520}\pi^2 + \tfrac{4065915400751}{191854512144}\right)\left(\frac{\alpha_s(\mu_0)}{\pi}\right)^3$$
$$+ \cdots + \mathcal{O}\left(\frac{\Lambda_{\overline{\text{MS}}}}{m_b}\right). \quad (5.48)$$

Table 5.1: The perturbative series for f_{B^*}/f_B

L	$-c_L/4^L$			NL/L	$-c_L\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L$
1	0.67	0.67			
2	4.06	2.08			
3		29.17	47.26	-0.50	0.0406
4		333.26	902.44	-0.33	0.0736
5		6342.19	18699.42	-0.26	0.1450
6		128998.30	449431.53	-0.21	0.3312

The asymptotics (5.31) becomes

$$c_{n+1}^{(m/\hat{m})} = \frac{16}{3} N_0\, n! \left(\frac{50}{3}\right)^n \left(\frac{50}{3}n\right)^{\frac{231}{625}} \times \left[1 + \frac{688161953}{1653125000}\frac{1}{n} + \left(-\frac{880261}{7187500}\zeta_3 + \frac{4}{625}\pi^2\log 2 + \frac{4}{625}\pi^2 + \frac{8332134087653830381}{49190800781250000000}\right)\frac{1}{n^2}\right]. \tag{5.49}$$

It depends on just one normalization constant N_0. In the large-β_0 limit, from (5.10) one obtains [99]

$$S(u) = 6C_F \left[\frac{\Gamma(u)\Gamma(1-2u)}{\Gamma(3-u)}(1-u) - \frac{1}{2u}\right], \tag{5.50}$$

and

$$\begin{aligned}
c_1^{(m/\hat{m})} &= \tfrac{3}{2}C_F\,, \quad c_2^{(m/\hat{m})} = C_F\left(\pi^2 + \tfrac{3}{4}\right)\beta_0\,,\\
c_3^{(m/\hat{m})} &= C_F\left(12\zeta_3 + \pi^2 + \tfrac{3}{4}\right)\beta_0^2\,,\\
c_4^{(m/\hat{m})} &= 3C_F\left(6\zeta_3 + \tfrac{3}{5}\pi^4 + \tfrac{1}{2}\pi^2 + \tfrac{3}{8}\right)\beta_0^3\,,\\
c_5^{(m/\hat{m})} &= 3C_F\left(144\zeta_5 + 16\pi^2\zeta_3 + 12\zeta_3 + \tfrac{6}{5}\pi^4 + \pi^2 + \tfrac{3}{4}\right)\beta_0^4\,,\\
c_6^{(m/\hat{m})} &= C_F\left(720\zeta_3^2 + 1080\zeta_5 + 120\pi^2\zeta_3 + 90\zeta_3 + \tfrac{244}{21}\pi^6 \right.\\
&\qquad \left. + 9\pi^4 + \tfrac{15}{2}\pi^2 + \tfrac{45}{8}\right)\beta_0^5\,, \ldots
\end{aligned}$$

(the terms with the highest powers of β_0 in the Appendix A are reproduced). Numerical results are shown in Table 5.2. For L=3, the $1/(L-1)$ expansion (5.49) seems to converge well; comparison with the exact 3-loop result from Appendix A suggests that the normalization factor N_0 is smaller that its large-β_0 value 1, namely, $N_0 \sim 0.27$. This conclusion is in a qualitative agreement with the estimate , especially if the problematic 3-loop correction in it is omitted. Finally, the last ratio (5.37) considered is

$$\frac{\hat{f}^T_{B^*}}{f_{B^*}} = 1 - \tfrac{707}{3174}\frac{\alpha_s(\mu_0)}{\pi} - \left(\tfrac{77}{138}\zeta_3 + \tfrac{1}{27}\pi^2\log 2 + \tfrac{1}{9}\pi^2 + \tfrac{1380868721}{1088021808}\right)\left(\frac{\alpha_s(\mu_0)}{\pi}\right)^2 + \cdots + \mathcal{O}\left(\frac{\Lambda_{\overline{\rm MS}}}{m_b}\right). \tag{5.51}$$

Table 5.2: The perturbative series for $m/\hat{m}$

L	$c_L/4^L$			NL/L	$c_L\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L$
1	0.84	0.5			
2	4.56	7.37			
3	54.97	36.23	947.38	0.21	0.1755
4		641.71	9990.14	0.14	0.2283
5		9062.48	150271.92	0.10	0.3899
6		206941.30	2925923.81	0.08	0.8225

The asymptotics (5.40) becomes

$$c_{n+1} = -\frac{16}{9}N_0\, n!\left(\frac{50}{3}\right)^n\left(\frac{50}{3}n\right)^{\frac{231}{625}}\left(1-\frac{513268907}{153125000}\frac{1}{n}+\cdots\right). \tag{5.52}$$

It also depends on just one normalization constant N_0. In the large-β_0 limit, from (5.10), $S(u)$ differs from (5.50) by the factor $-\frac{1}{3}$ (this reproduces the terms with the highest powers of β_0 in (5.40)). Numerical results are shown in Table 5.3.

Table 5.3: The perturbative series for $\hat{f}^T_{B^*}/f_{B^*}$

L	$-c_L/4^L$			NL/L	$-c_L\left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L$
1	0.22	0.17			
2	1.95	2.46			
3		12.08	−38.13	−1.68	−0.0327
4		213.90	−96.08	−1.12	−0.0076
5		3020.83	2459.47	−0.84	0.0191
6		68980.43	113207.40	−0.67	0.0834

Neglecting subleading α_s corrections, then, from (5.38) and (5.29), one obtains:

$$\hat{f}^T_{B^*}/f_{B^*} = \left(\hat{f}^P_B/f_B\right)^{-1/3}. \tag{5.53}$$

This equality also holds at the first order in $1/\beta_0$, to all orders of α_s. Therefore, the ratio of the perturbative coefficients (5.52) and (5.49) is $-\frac{1}{3}$, up to corrections suppressed by $1/n$ and $1/\beta_0$. Similarly, neglecting subleading α_s corrections, including those suppressed by $[\alpha_s/(4\pi)]^{\gamma_{m0}/(2\beta_0)}$, then, from (5.42) and (5.29), one obtains:

$$f_{B^*}/f_B = \left(\hat{f}^P_B/f_B\right)^{-\alpha}, \quad \alpha = -\frac{2}{3}\left(1-\frac{\gamma_0^m}{\gamma_{m0}}\right) = \frac{14}{27}.$$

Therefore, the leading asymptotics of the perturbative series for f_{B^*}/f_B (5.46) and $m/\hat{m}$ (5.49) are related by

$$f_{B^*}/f_B = (m/\hat{m})^{-\alpha}. \tag{5.54}$$

The term with N_2 in (5.46) violating this relation is suppressed not only by $(2\beta_0 n)^{-9/25}$, but also by a small numerical factor $\frac{2}{7}$. This approximate relation was first noted empirically at the 2-loop level in [50], with the exponent $\alpha = \frac{1}{2}$, which is very close to 14/27.

Let's summarize our main results.

1. Behaviour of the Borel images of perturbative series near the leading singularity $u = \frac{1}{2}$ for the matching coefficients (5.25), (5.35), and for the ratios $m/\hat{m}$ (5.30), $\hat{f}^T_{B^*}/f_{B^*}$ (5.39), and f_{B^*}/f_B (5.43) has been found. The powers of $\frac{1}{2} - u$ are exact; further corrections are suppressed by positive integer powers of $\frac{1}{2} - u$. The normalization factors $N_{0,1,2}$ cannot be found within this approach; they are some unknown numbers of order unity. Logarithmics branching is a new feature of this problem; it follows from the fact that the anomalous dimensions matrices cannot be diagonalized.

2. Asymptotics of perturbative coefficients c_L at $L \gg 1$ for the matching coefficients (5.27), (5.36), and for the same ratios (5.31), (5.40), (5.44) have been found. The powers of $n = L - 1$ are exact; further corrections are suppressed by positive integer powers of $1/n$. Logarithmic terms follow from the same property of the anomalous dimensions.

Chapter 6

Factorization at Subleading Order for Inclusive Decays

Contents

6.1 Introduction

Inclusive decays are defined as the decay of a particle into the sum of all possible final states with a given set of global quantum numbers. This class of decays is usually approximate to be the partonic decay. The theoretical study of these decays has two main advantages. First, the effects of the initial bound states, like the Fermi motion of the heavy quark, are treated in a systematic way using the Heavy Quark

Expansion. Second, the final state is a sum over all possible hadronic states and all the properties of the individual hadrons are eliminated. The latter feature is related to the so-called quark-hadron duality and is based on the fact that cross sections and decay rates are calculable in QCD after a smearing or averaging procedure has been applied [134]. From a practical point of view, the validity of this assumption relies on the convergence of the Operator Product Expansion (OPE) required to deal with these processes. However, it is important to stress that quark-hadron duality has not yet been derived from first principles, but, still, it is a necessary assumption for many applications of QCD.

For inclusive decays, an expansion for the rates is obtained by an approach similar to the one known from deep inelastic scattering [54–56, 59, 135]. The first step consists of an operator product expansion (OPE) which yields an infinite sum of operators with increasing dimension. The dimensions of the operators are compensated by inverse powers of a large scale, which is in general of the order of the heavy mass scale. The decay probability is then given as forward matrix elements of these operators between the state of the decaying heavy hadron; these matrix elements still have a mass dependence, which may be extracted in terms of a $1/m_Q$ expansion using HQET, as in exclusive decays.

Applying this idea to the energy spectrum, the relevant expansion parameter is not $1/m_Q$ but $1/(m_Q - 2E_\ell) \equiv 1/(m_Q\lambda^2)$ [62]. In almost all the phase space, $\lambda \sim 1$, and a $1/m_Q$ expansion is performed.

However, for inclusive decays into light particles, experimental cuts on the energy of the lepton pair are necessary to suppress the large background signal from charm production forcing the kinematics into the end point region of the spectrum, $\lambda \sim \Lambda_{QCD}/m_Q$, where the final state hadron carries large energy $E_H \sim m_Q$ but small invariant mass $s_H \sim m_Q\Lambda_{QCD}$. Therefore, the OPE breaks down. This can be solved by a resummation of the more singular terms of the OPE, or by performing a twist expansion in terms of non-local operators of non-perturbative character evaluated on the light cone.

In addition, in the end point region, the perturbative short distance corrections are more complicated than most of the cases where a heavy quark expansion in terms of local operators can be carried out, since involve three widely separate mass scales: m_Q (Hard), $\sqrt{m_Q\Lambda_{QCD}}$ (collinear) and Λ_{QCD} (soft). As usual, perturbation theory generates logarithms of theses ratios that have to be resummed.

A systematic treatment can be performed in the end point region for inclusive decays into light particle using effective field theories, along with a two-step matching procedure. First, matching QCD to SCET, where the hard short distance fluctuations are integrated out, and second, matching onto HQET where the collinear degrees of freedom are integrated out after a decoupling of soft and collinear degrees of freedom has been performed by a field redefinition.

A study at leading order, including radiative corrections shows that differential decays can be written in terms of $d\Gamma = H \otimes J \otimes S$ [16, 64–66], where H is the hard kernel accounting for the hard fluctuation of order m_Q, J is a Jet function, of the collinear scale $\sqrt{m_Q\Lambda_{QCD}}$, S is the leading shape function and $\otimes$ means a convolution integral.

In order to extract V_{ub} with a good accuracy, a study of power suppress contributions is needed. A tree analysis of them has been performed previously in [63,67–69]. Here the first step toward a systematic study of the subleading terms using SCET is presented at tree level. New shape functions not considered previously appear. Moreover, it will be shown that the factorization formulae beyond leading order hold.

This chapter will be focused on the study of the endpoint region of the spectrum. First, the techniques to deal with the spectrum decays will be presented, which will be written in terms of forward scattering matrices of the decaying particles. It will be shown that the normal OPE is not valid in the endpoint region and a distinction among different kinematical regions will be necessary where different effective field theories are required. In the next sections, the factorization for inclusive decays up to $1/m_Q$ will be presented in three steps, factorization of the hard fluctuations, decoupling of the soft-collinear degrees of freedom and factorization of these degrees of freedom.

6.2 Energy Spectrum

The starting point for a heavy flavor electroweak decay into light particles is the electroweak effective Hamiltonian. In a general way, it can be written as:

$$\mathcal{H}_{eff} = \frac{G_F}{\sqrt{2}} L J \tag{6.1}$$

G_F is the Fermi constant where is encode the matrix element of the CKM matrix, L is the non-hadronic current, and can be a leptonic pair or radiative products whereas J is a generic hadronic current of the form

$$J(x) \equiv \overline{q}(x)\Gamma Q(x) \tag{6.2}$$

being $Q(x)$ and $q(x)$ the heavy and the light quark respectively. The differential decay mediated by these currents can be written as:

$$d\Gamma = \frac{1}{2M_B} d[P.S.]_{pert}\langle 0|L^\dagger|X\rangle\langle X|L|0\rangle W(q) \tag{6.3}$$

where $|X\rangle$ is the non-hadronic decay product and W(q) is

$$W(q) = \sum_X d[P.S.]_{hadr}(2\pi)^4\delta^4(P_B - P_X - q)\langle B|J^\dagger|X\rangle\langle X|J|B\rangle \tag{6.4}$$

the sum runs over all possible hadronic final states $|X\rangle$. The matrix element of L can be calculated perturbatively, whereas $W(q)$ is related to the forward scattering amplitude

$$W(q) = -2\,\mathrm{ImT(q)} \tag{6.5}$$

where

$$T(q) \equiv i\int d^4x\, e^{-iqx}\langle B|T\left(J^\dagger(x)J(0)\right)|B\rangle \tag{6.6}$$

via the optical theorem. $T(q)$ is the hadronic tensor, which receives contributions from widely separated energy and distance scales. In terms of quarks the correlator reads as:

$$T(q) = i \int d^4x \; e^{iq\cdot x} \langle B(v)| T \left(\bar{Q}(x) \Gamma^\dagger \; q(x) \; \bar{q}(0) \Gamma Q(0) \right) |B(v)\rangle \tag{6.7}$$

The matrix element contains a large scale, the mass of the heavy quark m_Q. This scale can be made explicit by a phase redefinition of the heavy quark field:

$$Q(x) = e^{-im_Q v\cdot x} Q_v(x), \tag{6.8}$$

This leads to

$$T(q) = i \int d^4x \; e^{iQ\cdot x} \langle B(v)| T \left(\bar{Q}_v(x) \Gamma^\dagger \; q(x) \; \bar{q}(0) \Gamma Q_v(0) \right) |B(v)\rangle \tag{6.9}$$

where $Q = m_q v - q$ corresponds to the energy of the non-hadronic products.

As long as in euclidian space $Q^2 \gg 0$, the time–ordered product can be expanded by performing an OPE [43, 53, 59]

$$T(q) = \sum C_n(Q;\mu) \langle O_n(\mu) \rangle \tag{6.10}$$

where $\langle \ldots \rangle$ means matrix elements and it has been assumed that the analytical continuation from Euclidean space to Minkowski space is not an issue. Here, $C_n(Q;\mu)$ are the Wilson coefficients accounting for the short distance effects. The sum runs over all possible Lorentz and gauge invariant local operators compatible with the quantum numbers of the system. Performing an OPE expansion for $B \to X_s + \gamma$, one gets for the photon spectrum [56]:

$$\frac{d\Gamma}{dE_\gamma} \propto \sum_n a_n \delta^{(n)}(1-y) + (\text{regular terms at y=1}) \tag{6.11}$$

For $B \to X_u l \bar{\nu}_l$, the lepton spectrum looks like [60]:

$$\frac{d\Gamma}{dE_l} \propto \sum_n b_n \theta^{(n)}(1-y) + (\text{regular terms at y=1}) \tag{6.12}$$

where the subscript n in the distributions functions means the n-th derivative, whose coefficients are given by a_n and b_n. $y = 2E_{(\gamma,l)}/m_b$ is the normalized energy of the outgoing particle. Therefore, in the endpoint region $y \to 1$, one has an infinite series of divergent terms.

To understand the origin of this divergent behavior, one can study the shape of the correlator function at tree level. Doing so, one becomes aware of three different kinematical regions where a different theory is required, making clear the failures of the OPE expansion for the endpoint region.

At tree level, the leading contribution to the correlator is obtained by contracting the light-quarks fields [62]:

$$T(q) = i \int \mathrm{d}^4x \, e^{iQ\cdot x} \, \langle B(v)| \, T \left\{ \bar{b}_v(x) \, \Gamma^\dagger \, S_q(x,0) \, \Gamma \, b_v(0) \right\} |B(v)\rangle \tag{6.13}$$

Here, $S_q(x,0)$ is the propagator of the q-quark. At tree level the momentum of the light quark is $p_q = p_b - q = Q + k$ where k is the residual momenta of the heavy quark $k = p_b - m_b v$. Therefore,

$$
\begin{aligned}
S_q(Q) &= \frac{1}{\not{Q} + \not{k} + i\epsilon} \\
&= (\not{Q} + \not{k})\,\frac{1}{(Q^2 + 2Q\cdot k + k^2 + i\epsilon)}\,. \qquad (6.14)
\end{aligned}
$$

In (6.14) , ϵ denotes, as usual, an infinitesimal positive number and gives the prescription to deal with poles or branch-cut singularities. Since the residual momentum associated with the interactions of the heavy quark with light degrees of freedom is of order Λ_{QCD}, the matrix elements are of the order:

$$
\begin{aligned}
\langle\, i\hat{k}\,\rangle &\sim O(\Lambda_{QCD}), \\
\langle\, 2k\cdot Q\,\rangle &\sim O(\Lambda_{QCD}\, E_Q), \\
\langle\, k^2\,\rangle &\sim O(\Lambda^2_{QCD}), \qquad (6.15)
\end{aligned}
$$

where E_Q, of order m_Q and, $Q = m_Q\cdot v - q$ would correspond to the energy and the momentum of the light quark considering the heavy quark on-shell. Depending of the value of its invariant mass, one can distinguish among three kinematical regions where a different OPE or theory is needed.

$$
\begin{aligned}
i) &\qquad Q^2 \sim O(E_Q^2), \\
ii) &\qquad Q^2 \sim O(\Lambda^2_{QCD}\,), \\
iii) &\qquad Q^2 \sim O(\Lambda_{QCD}\, E_Q). \qquad (6.16)
\end{aligned}
$$

Let discuss these regions.

General Kinematical Regions

i) The first one corresponds to the energetic hadronic jet with energy close to the decaying particle m_Q and with a large invariant mass. In this situation, all the terms with k can be neglected and it is legitimate to expand the correlator in a series of local operators multiplied by coefficients functions that contain inverse powers of Q, recovering the free quark decay model up to small non-perturbative corrections, like the Fermi motion of the heavy quarks. A higher accuracy is reached considering higher order terms of the expansion[1]. Over the most of the phase space $Q^2 \sim m_Q^2$, and thus, the expansion reduces to an expansion in powers of Λ_{QCD}/m_Q yielding the presented energy spectrum in (6.11,6.12).

ii) For $Q^2 \sim \Lambda^2_{QCD}$, the term with k becomes dominant in the denominator of the propagator and no term can be neglected. This is the resonance region where the dynamics is dominated by the emission and the consequent decay of few

[1] It is clear that a consistent inclusion of the $1/m_Q$ corrections involves also the expansion of the heavy quark field $b(x)$ into the effective quark field $h_v(x)$ up to the required order.

resonances and can be deal in a systematic way in the framework of $SCET_I$ and $SCET_{II}$, in the lattice or using quark models. The fact that the resonance region is parametrically smaller than the endpoint region allows one to neglect it in the study of the endpoint region [62].

iii) For the endpoint region, the final jet has a relative small invariant mass $Q^2 \sim \Lambda_{QCD} E_Q$ but $E_Q \sim m_Q$. It is not possible to neglect the term linear with k in the denominator. Then, the expansion performed in *i*) breaks down in an infinite series of singular terms. This can be resummed in terms of non-local operators. But the formalism is much simple when presented in using the language of soft-collinear effective theory.

6.3 End-point Region Kinematics

The kinematics of the end-point region becomes more clear when presented for the radiative case $B \to X_s \gamma$ at the partonic level. It is convenient to introduce two light-like vectors $n_+ = (1, 0, 0-, 1)$ and $n_- = (1, 0, 0, 1)$ with $n_-^2 = n_+^2 = 0$ and $n_- \cdot n_+ = 2$. Any 4-vector can be expanded in terms of the light-cone vectors:

$$p^\mu = (n_- p)\frac{n_+^\mu}{2} + (n_+ p)\frac{n_-^\mu}{2} + p_\perp^\mu \equiv (p_+, p_-, p_\perp) \tag{6.17}$$

The b quark is defined to be at rest $p_b = m_b \cdot v$ with $v^\mu = (1, 0, 0, 0)) \equiv m_b/2(1, 1, 0)$. For the radiative product, $q^2 = 0$ and defines one of the light-cone directions $q = m_b/2(x, 0, 0)$, in the End point region $x \to 1$. Finally, the final s quark

$$p_s = Q = p_b - q = \frac{m_b}{2}(1 - x, 1, 0) = \frac{m_b}{2}(\lambda^2, 1, 0) \tag{6.18}$$

Moreover, the End point region is defined to scale as: $p_s^2 = m_b \Lambda_{QCD}$ and then $\lambda = \sqrt{\Lambda_{QCD}/m_b}$. One can generalize this result for the hadronic jet either for the semileptonic and for the radiative decay to:

$$p_J^2 = m_b \lambda^2 = p_- p_+ + p_\perp^2 \to p_J = m_b(\lambda^2, 1, \lambda) \tag{6.19}$$

However, by a choice of coordinates, the perpendicular component always can be set to zero. The important thing to remark is that this is exactly the power counting that make use SCET. Therefore, one can use the machinery of SCET described in Chapter 3 in order to deal with these decays in the end point region.

6.4 Factorization of the Hadronic Tensor within SCET

In this section the correlator will be calculated using SCET. At leading order the hadronic tensor can be written generally as $T \sim H \cdot J \otimes S$ with H, the hard kernels, which picks the hard scale, J, a jet function of the collinear scale and S, a shape functions which carries the non-perturbative information of the B meson-state.

Here, the Hadronic tensor beyond leading order will be presented. It will be shown that the same pattern is obtained.

This will be achieved in three steps. Below the scale $\mu \sim m_b$, the current can be expanded using SCET. In addition, one has to take into account corrections coming from the matrix elements in term of insertion of Lagrangians. In this first step the hard-collinear interaction will be integrated out.

Second, by a field redefinition the collinear and soft degrees of freedom are decoupled at the level of Lagrangian. The hadronic tensor is sandwiched between B meson states, which only contains soft degrees of freedom. Therefore, the collinear fields have to be created and annihilated in the vacuum. These involves scales of $\sqrt{\Lambda_{QCD}}m_b$ and can be calculated using perturbative methods.

Third, matching onto HQET the collinear degrees of freedom are integrated out. One is left only with the soft degrees of freedom, which contain the non-perturbative physics of the system. These will be parametrized in terms of scalar "shape functions".

Although the formalism is general, the calculation presented here and results in Appendix B are at tree level and in the light cone gauge. This will greatly simplify our analysis.

Under the scale $\mu \sim m_b$ the QCD currents are expanded as:

$$(\bar{Q}\Gamma_i q)(x) = \sum_k \int d\hat{s}_1 d\hat{s}_2 \dots d\hat{s}_n \sum_j \hat{C}_{ij}^{(k)}(\hat{s}_1, \hat{s}_2, \dots \hat{s}_n) J_j^{(k)}(\hat{s}_1, \hat{s}_2, \dots \hat{s}_n; x) \quad (6.20)$$

$C_{ij}^{(k)}$ are the Wilson coefficients and factorize the hard scale. The suppress λ terms at one loop were calculated in [136, 137] This factorization can be performed at all orders in α_s and can be done at all order in λ, J are the SCET currents. The i stands for a different type of Gamma matrices, j accounts for the mixing of the operator beyond tree level, k, means the order of the expansion, being 0 the leading order and ds_i are the convoluted variable in the hard scale. At tree level this simplifies to:

$$\left[\bar{\psi}(x)\,\Gamma\,Q(x)\right]_{\rm QCD} = e^{-imv\cdot x}\left\{J^{(A0)} + J^{(A1)} + J^{(A2)} + J^{(B1)} + J^{(B2)}\right\} \quad (6.21)$$

with

$$\begin{aligned}
J^{(A0)} &= \bar{\xi}\,\Gamma W_c h_v, \qquad (6.22)\\
J^{(A1)} &= \bar{\xi}\,\Gamma W_c\, x_{\perp\mu} D_{\rm s}^{\mu} h_v - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma W_c h_v,\\
J^{(A2)} &= \bar{\xi}\,\Gamma W_c\left(\frac{1}{2}n_- x\, n_+ D_{\rm s} h_v + \frac{1}{2}x_{\mu\perp}x_{\nu\perp}D_{\rm s}^{\mu}D_{\rm s}^{\nu}h_v + \frac{i\not{D}_{\rm us}}{2m}h_v\right)\\
&- \bar{\xi}\,\Gamma\frac{1}{in_+D_c}\left[in_- D\, W_c - W_c\, in_- D_{\rm s}\right]h_v - \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma W_c\, x_{\perp\mu}D_{\rm s}^{\mu}h_v,\\
J^{(B1)} &= -\bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\left[i\not{D}_{\perp c}W_c\right]h_v,\\
J^{(B2)} &= -\bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\left[i\not{D}_{\perp c}W_c\right]x_{\mu\perp}D_{\rm s}^{\mu}h_v - \bar{\xi}\,\Gamma\frac{\not{n}_-}{2m}\left[in_- D\, W_c - W_c\, in_- D_{\rm s}\right]h_v\\
&- \bar{\xi}\,\Gamma\frac{1}{in_+D_c}\left[\frac{i\not{D}_{\perp c}i\not{D}_{\perp c}}{m}W_c\right]h_v + \bar{\xi}\, i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}\frac{\not{n}_+}{2}\Gamma\frac{\not{n}_-}{2m}\left[i\not{D}_{\perp c}W_c\right]h_v,
\end{aligned}$$

where all the hard functions are unity in momentum space. Moreover, from the time ordered products and matrix elements, one has to consider insertions of the HQET Lagrangian:

$$\mathcal{L} = \bar{h}_v iv\cdot Dh_v + \frac{1}{2m}\left[O_k + C_m(\mu)O_m(\mu)\right] + \mathcal{O}(1/m^2)\,, \qquad (6.23)$$
$$O_k = -\bar{h}_v D_\perp^2 h_v\,, \quad O_m = \frac{1}{2}\bar{h}_v G_{\alpha\beta}\sigma^{\alpha\beta}h_v\,,$$

h_v is a soft field and the derivatives $D = \partial - igA_s$ only contains soft degrees of freedom, C_m is the chromomagnetic Wilson coefficient and contains the hard scale and together with $\hat{C}_{ij}(\mu)$ collect all the hard interactions. Second, by the SCET Lagrangian:

$$\mathcal{L} = \bar{\xi}\left(in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi + \bar{q}(x)\,i\not{D}_{\rm s}(x)q(x) + \mathcal{L}_\xi^{(1)} + \mathcal{L}_\xi^{(2)} + \mathcal{L}_{\xi q}^{(1)} + \mathcal{L}_{\xi q}^{(2)}, \qquad (6.24)$$

where the power-suppressed interaction terms are given by

$$\mathcal{L}_\xi^{(1)} = \bar{\xi}\left(x_\perp^\mu n_-^\nu\, W_c\, gF^{\rm s}_{\mu\nu}W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi, \qquad (6.25)$$
$$\mathcal{L}_\xi^{(2)} = \frac{1}{2}\bar{\xi}\left((n_-x)\,n_+^\mu n_-^\nu\, W_c\, gF^{\rm s}_{\mu\nu}W_c^\dagger + x_\perp^\mu x_{\perp\rho}n_-^\nu W_c[D_{\rm s}^\rho, gF^{\rm s}_{\mu\nu}]W_c^\dagger\right)\frac{\not{n}_+}{2}\,\xi$$
$$+\frac{1}{2}\bar{\xi}\left(i\not{D}_{\perp c}\frac{1}{in_+D_c}\,x_\perp^\mu\gamma_\perp^\nu\, W_c\, gF^{\rm s}_{\mu\nu}W_c^\dagger + x_\perp^\mu\gamma_\perp^\nu\, W_c\, gF^{\rm s}_{\mu\nu}W_c^\dagger\,\frac{1}{in_+D_c}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi,$$
$$\mathcal{L}_{\xi q}^{(1)} = \bar{q}\,W_c^\dagger i\not{D}_{\perp c}\,\xi - \bar{\xi}\,i\overleftarrow{\not{D}}_{\perp c}W_c\,q,$$
$$\mathcal{L}_{\xi q}^{(2)} = \bar{q}\,W_c^\dagger\left(in_-D + i\not{D}_{\perp c}\,(in_+D_c)^{-1}\,i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi + \,\bar{q}\,x_{\perp\mu}\overleftarrow{D}_{\rm s}^\mu W_c^\dagger\,i\not{D}_{\perp c}\xi$$
$$-\,\bar{\xi}\,\frac{\not{n}_+}{2}\left(in_-\overleftarrow{D} + i\overleftarrow{\not{D}}_{\perp c}\left(in_+\overleftarrow{D}_c\right)^{-1}i\overleftarrow{\not{D}}_{\perp c}\right)W_c\,q - \,\bar{\xi}\,i\overleftarrow{\not{D}}_{\perp c}W_c\,x_{\perp\mu}D_{\rm s}^\mu q.$$

Hence the hadronic tensor can be written generically by:

$$T = \sum H\otimes T^{eff} \qquad (6.26)$$

H is the hard function coming from the Wilson coefficients of the SCET currents and the HQET Lagrangian and T^{eff} is the time-ordered product of SCET Currents with Lagrangians inserted involving collinear and the soft degrees of freedom which are coupled by (6.24). At tree level, H are delta functions and the convolution becomes a simple product. T^{eff} at tree level is:

$$T^{eff} = i\frac{1}{n}\sum\int d^4x\int d^4y_1\ldots d^4y_n J_j^{(A,Bi)\dagger}(x)\,i\mathcal{L}^k(y_1)\ldots\,i\mathcal{L}^k(y_n)J_m^{(A,Bl)\dagger}(0) \quad (6.27)$$

The contribution at leading order will come from products of:

- $J^{(A0)\dagger}(x)J^{(A0)}(0)$

At the first subleading order from:

- $J^{(A1)\dagger}(x)J^{(A0)}(0)+$ symmetric
- $J^{(A0)\dagger}(x)\mathcal{L}^{1}(y)J^{(A0)}(0)$

In addition at every order one can insert an arbitrary number of leading order Lagrangians which couple soft and collinear degrees of freedom and prevent us to factorize the soft from the collinear fluctuations.

In next section, a field redefinition that decouples the soft degrees of freedom from the leading order Lagrangian will be given.

6.4.1 Decoupling Soft Degrees of Freedom

At every order in the hadronic tensor, one can have an arbitrary number of insertions of the leading order Lagrangian,

$$\mathcal{L}_c^{(0)} = \bar{\xi}\left(in_-D + i\not{D}_{\perp c}\frac{1}{in_+D_c}\, i\not{D}_{\perp c}\right)\frac{\not{n}_+}{2}\,\xi, \tag{6.28}$$

where $in_-D = in_-\partial + gn_-A_c + n_-A_s$ couples soft and collinear degrees of freedom. However, this coupling only appears through the n_- light cone vector. One can remove this interaction by a field redefinition,

$$\xi(x) = [Y\xi^{(0)}](x), \quad A_c = [YA_c^{(0)}Y^\dagger](x), \quad \text{and} \quad W_c = (YW_c^{(0)}Y^\dagger)(x). \tag{6.29}$$

with $Y(x)$ the Wilson line:

$$Y(x) \equiv Pexp\left(i\,g\int_{-\infty}^{0} ds\, n_-\cdot A_s(x_- + sn_-)\right) \tag{6.30}$$

which satisfies $[in_-DY] = 0$. The Lagrangian in terms of the new fields:

$$\mathcal{L}_c = \bar{\xi}^{(0)}\left(in_-D_c^{(0)} + i\not{D}_{c\perp}^{(0)}\frac{1}{in_+D_c^{(0)}}\, i\not{D}_{c\perp}^{(0)}\right)\frac{\not{n}_+}{2}\,\xi^{(0)} \tag{6.31}$$

where only collinear fields appear. The same occurs with the Yang Mill sector. Hence, the leading order effective Lagrangian will look as:

$$\mathcal{L}_{eff} = \underbrace{\mathcal{L}_c + \mathcal{L}_c^{YM}}_{\text{collinear}} + \underbrace{\mathcal{L}_s + \mathcal{L}_s^{YM} + \mathcal{L}_{HQET}}_{\text{soft}} + \dots \tag{6.32}$$

which factorizes soft and collinear interactions. The leading order currents in terms of the new fields are written as:

$$J^{(A0)} = [\bar{\xi}\,\Gamma W_c h_v](x), \rightarrow J^{(A0)} = [\bar{\xi}^{(0)}\Gamma W_c^{(0)} Y h_v](x) \tag{6.33}$$

The field redefinition complicates the form of the currents, but, now, $\bar{\xi}^{(0)}W_c^{(0)}$, the collinear fields, and $[Yh_v](x_-)$, the soft ones, are not coupled by the leading order Lagrangian and can be treated separately. Although, they are mixed by the first subleading Lagrangian even after the field redefinition,

$$\mathcal{L} = \bar{\xi}^{(0)}\left(x_\perp^\mu n_-^\nu\, W_c^{(0)}Y\, gF_{\mu\nu}^{s} Y^\dagger W_c^{(0)\dagger}\right)\frac{\not{n}_+}{2}\,\xi^{(0)}, \tag{6.34}$$

one can show order by order the factorization of the soft and collinear degrees of freedom at all orders.

6.4.2 Factorization at Leading Order in $1/m_b$

Inserting the leading order current, the effective correlators are:

$$T^{eff}(q) = i\int d^4x\, e^{iQx} \\ \langle B(v)|T\left[\bar{h}_v(x_-)W_c^\dagger(x)\Gamma^\dagger\xi(x)\bar{\xi}(0)\Gamma W_c(0)h_v(0)\right]|B(v)\rangle \tag{6.35}$$

Performing the field redefinition:

$$T^{eff}(q) = i\int d^4x\, e^{iQx} \\ \langle B(v)T\left(\left[\bar{h}_v Y W_c^{\dagger(0)}\Gamma^\dagger\xi^{(0)}\right](x)\left[\bar{\xi}^{(0)}\Gamma W_c^{(0)}Y^\dagger h_v\right](0)|B(v)\right)\rangle \tag{6.36}$$

The new collinear fields do not couple with soft fields at leading order, since the leading SCET Lagrangian does not couple them. The B-meson state by definition does not contain collinear degrees of freedom since the b quark, inside of it, is supposed to be almost on-shell, and hence an interaction with a collinear particle would put it far off-shell $(p_b+p_c)^2 > m_b^2$. Therefore, the collinear degrees of freedom have to be created and annihilated with the vacuum. Leaving the color and the spinor indices open, the correlator gives:

$$T^{eff}(q) = i\int d^4x\, e^{iQx} \\ \langle B(v)\left[\bar{h}_v Y\right]_{a\alpha}(x_-)\left[Y^\dagger h_v\right]_{b\beta}(0)|B(v)\rangle \times \\ \langle\Omega|T\left(\left[W_c^{\dagger(0)}\xi^{(0)}\right]_{a\delta}(x)\left[\bar{\xi}^{(0)}W_c^{(0)}\right]_{b\gamma}(0)\right)|\Omega\rangle\Gamma_{\alpha\delta}\Gamma^\dagger_{\gamma\beta} \tag{6.37}$$

The collinear correlator carries the scales $\sqrt{\Lambda_{QCD}m_b}$ and can be evaluated perturbatively:

$$\langle\Omega|T\left(\left[W_c^{\dagger(0)}\xi^{(0)}\right]_{a\delta}(x)\left[\bar{\xi}^{(0)}W_c^{(0)}\right]_{b\gamma}(0)\right)|\Omega\rangle = \\ \delta_{ab}\int\frac{d^4l}{(2\pi)^4}\,e^{-il\cdot x}\,i\frac{\not{n}_{-\delta\gamma}}{2}\,\hat{J}_0(l^2) \tag{6.38}$$

where $\hat{J}(l^2)$ defines the Jet function. At tree level in the light cone, $W_c = 1$,

$$\hat{J}_0(l^2) = \frac{1}{l_+ + \frac{l_\perp^2}{l_-} + i\epsilon}. \tag{6.39}$$

Corrections to this order in α_s are calculated by inserting the collinear part of the leading order Lagrangian, if one inserts subleading order Lagrangian one would have as well a suppression in λ in addition to the perturbative one. At one loop, the collinear jet function has been calculated in [64, 65]. The correlator is given by:

$$T^{eff}(q) = -\delta_{ab}\Gamma_{\alpha\delta}\frac{\not{n}_{-\delta\gamma}}{2}\Gamma^\dagger_{\gamma\beta}\int d^4x\,\frac{d^4l}{(2\pi)^4}\,e^{ix(Q-l))}\,\hat{J}_0(l^2) \\ \langle B(v)\left[\bar{h}_v Y\right]_{a\alpha}(x_-)\left[Y^\dagger h_v\right]_{b\beta}(0)|B(v)\rangle \tag{6.40}$$

For the radiative decay the exponential factor is:

$$x(m_b v-(q+l)=(m_b-l_-)x_+/2+(m_b-(q+l)_+)x_-/2+(q+l)_\perp. \qquad (6.41)$$

Moreover, since the soft field only depends on the x_- variable, rewriting the integration variable in terms of light cone coordinates,($d^4Q=1/2dQ_-dQ_+dQ_\perp$), one obtains delta functions from the $dx_\perp\ dx_+$ integration, which can be used to integrate dl_- and $dl_\perp$ in turn. The correlator gives:

$$\begin{aligned}T^{eff}(q) &= -\int dx_-\frac{dl_+}{2(2\pi)}\,e^{ix_-/2(Q-l)_+)}\ \hat{J}_0(l_+)\\ &\langle B(v)\left[\bar{h}_vY\right]_{a\alpha}(x_-)\delta_{ab}\Gamma_{\alpha\delta}\frac{\not{n}_{-\delta\gamma}}{2}\Gamma^\dagger_{\gamma\beta}\left[Y^\dagger h_v\right]_{b\beta}(0)|B(v)\rangle \end{aligned}\qquad (6.42)$$

The heavy quark operator can be written as:

$$\begin{aligned}\left[\bar{h}_vY\right](x_-)\bar{\Gamma}\left[Y^{(0)}h_v\right](0) &= \left[\bar{h}_vY\right](0)\bar{\Gamma}e^{\overleftarrow{\partial}_+x_-}\left[Y^{(0)}h_v\right](0)=\bar{h}_{v\alpha}\bar{\Gamma}e^{\overleftarrow{D}_+x_-}h_{v\beta}\\ &= \int dk_+e^{-(i/2)k_+x_-}\ h_v\bar{\Gamma}\delta(k_++iD_+)h_v \end{aligned}\qquad (6.43)$$

where the property of the Wilson line has been applied $[inDY]=0$ and the covariant derivative acts to the right after doing an integration by parts in the hadronic tensor. The matrix element using

$$P_+\Gamma P_+=\frac{1}{2}P_+\mathrm{Tr}(P_+\Gamma)-\frac{1}{2}s_\mu\mathrm{Tr}(s^\mu\Gamma)\equiv\frac{1}{2}\mathrm{Tr}\left[P^j\Gamma\right] \qquad (6.44)$$

where $s_\mu\equiv P_+\gamma_\mu\gamma_5P_+$ between heavy quark fields can be written as:

$$\begin{aligned}&\langle B(v)|T\left(\left[\bar{h}_vY\right]_{a\alpha}(x_-)\left[Y^{(0)}h_v\right]_{b\beta}(0)\right)|B(v)\rangle=\\ &\qquad\frac{\delta_{ab}}{N_c}\int dk_+e^{-(i/2)k_+x_-}\ S(k_+)\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\end{aligned}\qquad (6.45)$$

with

$$S(k_+)=\langle B(v)|\,\bar{h}_v\,\delta(k_++iD_+)\,h_v\,|B(v)\rangle \qquad (6.46)$$

the leading shape function, the odd part vanishes by parity invariance. Hence,

$$\begin{aligned}T^{eff}(q) &= -\int dx_-dk_+\frac{dl_+}{2(2\pi)}\,e^{ix_-/2(m_b-(q+l+k)_+)}\ \hat{J}_0(l_+)S(k_+)\times\\ &\delta_{ab}\frac{\delta_{ab}}{N_c}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\Gamma_{\alpha\delta}\frac{\not{n}_{-\delta\gamma}}{2}\Gamma^\dagger_{\gamma\beta}\end{aligned}\qquad (6.47)$$

Inserting this in the hadronic correlator function and integrating:

$$T^{eff}(q)=\frac{-1}{2}\int dk_+\,S(k_+)\,\hat{J}_0(m_b-(q+k)_+)\,\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \qquad (6.48)$$

Defining:

$$J_0(l^2)=\mathrm{Im}\hat{J}_0(l^2) \qquad (6.49)$$

The imaginary part of the correlators

$$\mathrm{Im}T^{eff}(q) = \frac{-1}{2}\int dk_+ S(k_+) J_0(m_b - (q+k)_+)\ \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{6.50}$$

including the hard kernel:

$$\mathrm{Im}T(q) = \frac{-1}{2}H(n_+Q)\int dk_+ S(k_+) J_0(m_b - (q+k)_+)\ \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{6.51}$$

where H contains the hard scale $Q_+ \sim m_b$, J_0 the collinear scale $\sim \sqrt{\Lambda_{QCD} m_b}$ and S the soft scale $k_+ \sim \Lambda_{QCD}$. One could use the renormalization group equation to run the scale down to soft scales performing the resummation of the large logs [64]. In the next Section the subleading terms will be investigated.

6.4.3 Beyond Leading Order

The fact that the leading order Lagrangian does not couple the soft and collinear degrees of freedom was relevant in order to show factorization at leading order. But, the first subleading Lagrangian couples them. This can break the factorization theorem at subleading order. It will be shown that order by order this factorization can be achieved at the price of obtaining more complicated sub-leading shape functions which accommodate soft degrees of freedom coming from the subleading Lagrangian and from the currents. The leading order correlator function with an insertion of the first subleading Lagrangian yields a subleading contribution,

$$\begin{aligned} T^{eff} &= i\int dx^4 dy^4\, e^{iQx}\langle \bar{B}(p)|T\left(J^{0\,\dagger}_{eff}(x) i\mathcal{L}^1(y)\, J^0_{eff}(0)\right)|\bar{B}(p)\rangle \\ &= i\int dx^4 dy^4\, e^{iQx}\langle \bar{B}(p)T\left([\bar{h}_v\, Y\Gamma^\dagger\, \xi^{(0)}](x)\right. \\ &\qquad \left.\bar{\xi}^{(0)}\ \left(y_\perp^\mu\, n_-^\nu\, Y^\dagger\, i\, g\, F^s_{\mu\nu}\, Y\right)\frac{\not{n}_+}{2}\xi^{(0)}](y)\,\Gamma Y^\dagger\, h_v](0)\right)|\bar{B}(p)\rangle \end{aligned} \tag{6.52}$$

where W_c is not written. At this order in the expansion, again, the redefined fields do not couple through the leading order Lagrangians, one has to insert sub-leading Lagrangians to do so. Therefore,

$$\begin{aligned} T^{eff} &= i\Gamma^\dagger_{\alpha\kappa}\frac{\not{n}_{+\gamma\sigma}}{2}\Gamma_{\delta\beta}\int dx^4 dy^4\, e^{iQx}\ \times \\ &\qquad \langle\Omega|T\left(\xi^{(0)}_{c\kappa}(x)\,[\bar{\xi}^{(0)}_{d\gamma}\, y^\mu_\perp \xi^{(0)}_{e\sigma}](y)\bar{\xi}^{(0)}_{f\delta}\,(0)\right)|\Omega\rangle\ \times \\ &\qquad \langle \bar{B}(p)|T\left([\bar{h}_v\, Y\,]_{a\alpha}(x_-)[n_-^\nu\, Y^\dagger i\, g\, F^s_{\mu\nu}\, Y]_{de}(y_-)[Y^\dagger\, h_v]_{b\beta}\,(0)\right)|\bar{B}(p)\rangle \end{aligned} \tag{6.53}$$

The collinear matrix element at tree level in the light cone gauge is,

$$\begin{aligned} &\langle\Omega|T\left(\xi^{(0)}_{c\kappa}(x)\,[\bar{\xi}^{(0)}_{d\gamma}\, y^\mu_\perp \xi^{(0)}_{e\sigma}](y)\bar{\xi}^{(0)}_{f\delta}\,(0)\right)|\Omega\rangle = \\ &\qquad - i\delta_{cd}\delta_{ef}\frac{\not{n}_{-\kappa\gamma}}{2}\frac{\not{n}_{-\sigma\delta}}{2}\int d^4p\, d^4l e^{-ipx}e^{-ily}\hat{J}^\mu(p^2, l^2) \end{aligned} \tag{6.54}$$

with

$$\hat{J}^{\mu}(p^2, l^2) = \hat{J}_0(p^2)\hat{J}_1^{\mu}(l^2), \tag{6.55}$$

and

$$\hat{J}_1^{\mu}(l^2) = -\frac{\partial}{\partial l_{\perp\mu}}\hat{J}_0(l^2). \tag{6.56}$$

In Appendix B.1 a list of the jet functions that appear at tree level can be found. The heavy quark operator can be written in analogy with (6.43):

$$\begin{aligned}\left[\bar{h}_v Y\right](x_-)\bar{\Gamma}\left[Y^\dagger\, i\, g\, F^s_{\mu\nu} Y\right](y_-)\left[Y^\dagger h_v\right](0) = \\ \int ds_+ dk_+ e^{-(i/2)s_+(x-y)_-} e^{-(i/2)k_+ y_-} O^4_{\mu\nu}(s_+, k_+)\end{aligned} \tag{6.57}$$

with

$$O^4_{\mu\nu}(s_+, k_+) = \bar{h}_v\, \bar{\Gamma}\delta(s_+ + iD^s_+)\ i\, g n^\nu_- F^s_{\mu\,\nu}\,\delta(k_+ + iD^s_+)\ h_v \tag{6.58}$$

Using the projector properties (6.44):

$$O^4_\mu(s_+, k_+) = \mathrm{Tr}\left[\frac{P^j}{2}\bar{\Gamma}\right] O^{4,j}_\mu \tag{6.59}$$

Taking matrix element of this operators, only the part related with the odd structure s^ρ of the matrices will contribute, since together with $F_{\mu\nu}$ it is possible to build an even operator. Generally, this soft matrix element between B-meson states satisfying all the symmetries, projecting into the heavy quark fields, and taking into account that the n_- and v vectors are available can be parametrized in terms of scalar shape functions:

$$\begin{aligned}\langle \bar{B}_v|\bar{h}_v(x_-)_{a\alpha} n^\nu_- i g F^s_{\mu\perp\nu}(z_-)_{cd} h_v(0)_{b\beta}|\bar{B}_v\rangle = \\ \frac{2T^A_{ba}T^A_{cd}}{N_c^2-1}\frac{1}{2}\left(\frac{1+\not{v}}{2}\gamma^{\rho\perp}\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}\frac{\epsilon^\perp_{\mu\rho}}{2} n_- v C_1(x_-, z_-)\end{aligned} \tag{6.60}$$

where the color and the spinor structure are defined to give the unity by contracting with $\delta_{ac}\delta_{db}$. $C_1(x_-, y_-)$ is a sub-leading scalar shape function and can be identified by:

$$\frac{\epsilon^\perp_{\mu\rho}}{2} n_- v C_1(s_+, k_+) = \langle B(v)|O^{4,2}_\mu(s_+, k_+)|B(v)\rangle \tag{6.61}$$

The soft fields only depend of the x_- and y_- coordinates and again the integration of $dx_\perp$ and dx_- can be performed in the collinear sector. Integrating, the hadronic correlator in terms of the scalar shape function is:

$$\begin{aligned}T^{eff}(q) = \frac{1}{2}\int ds_+\, dk_+ C_1(s_+, k_+)\hat{J}^\mu(Q^2 - s_+Q_-, Q^2 - k_+Q_-) \\ (n_- v)\frac{\epsilon^\perp_{\mu\rho}}{2} \times \mathrm{Tr}\left[s^\rho \Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right]\end{aligned} \tag{6.62}$$

Adding the hard-collinear interaction:

$$T(q) = H(n_-Q)\frac{1}{2}\int ds_+\, dk_+ C_1(s_+,k_+)\hat{J}^\mu(Q^2 - s_+Q_-, Q^2 - k_+Q_-)$$
$$(n_-v)\frac{\epsilon^\perp_{\mu\rho}}{2} \times \text{Tr}\left[s^\rho \Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{6.63}$$

where H again carries the hard scale, $\hat{J}^\mu$ the collinear scale and S_1 the soft degrees of freedom. Hence, a similar factorization formulae beyond leading order hold.

For the multiple expansion the soft fields depend only on one light-cone direction which ensures the convolution between soft and collinear fields along that direction. Two convolution variables appear at first subleading order when inserting the first subleading Lagrangian. At order λ^n, up to $n+1$ convolution variable can appear due to $(\mathcal{L}^1)^n$ insertions. The fact that the subleading Lagrangian mixed soft and collinear degrees of freedom has not spoilt the factorization formulation. At every order, the hadronic tensor can be factorized due to the factorization of the Leading order Lagrangian. However, new subleading, non-perturbative, shape functions appear at every order and might accommodate information both from the current and from the subleading Lagrangians. A classification of them must be done at all orders.

In the next Section, the calculation of the hadronic tensor up to order $1/m$ is performed at tree level.

6.5 Correlator at $\mathcal{O}(\lambda^2)$

In this section the correlator at $\mathcal{O}(\lambda^2)$ will be calculated at tree level and in the light cone, $W_c = 1$. Besides Eq. (6.63) the remaining contribution at order λ come from insertions of the first subleading currents:

$$J^{(A1)\dagger}(x)J^{(A0)}(0) + \text{symmetric}. \tag{6.64}$$

At $\lambda^2 \sim \Lambda_{QCD}/m_b$ the contribution will come from time order products of:

- $J^{(A2)\dagger}(x)J^{(A0)}(0)+$ symmetric
- $J^{(A1)\dagger}(x)J^{(A1)}(0)$
- $J^{(A0)\dagger}(x)\mathcal{L}^2(y)J^{(A0)}(0)$
- $J^{(A1)\dagger}(x)\mathcal{L}^1(y)J^{(A0)}(0)+$ symmetric
- $J^{(A1)\dagger}(x)\mathcal{L}^1(y)\mathcal{L}^1(y)J^{(A0)}(0)$

At tree level, the current needed at order $\mathcal{O}(\lambda)$ are:

$$J^{(A1)}_{(1)}(x) = \bar{\xi}\,\Gamma\, W_c Y^\dagger\, x_{\perp\mu}\, D^\mu_{us} h_v$$
$$J^{(A1)}_{(2)}(x) = -\bar{\xi}\, i\, \overleftarrow{\not{D}}_{\perp c}\,(in_+\overleftarrow{D})^{-1}\frac{\not{n}_+}{2}\,\Gamma\, W_c Y^\dagger\, h_v$$

At second order:

$$
\begin{aligned}
J^{(A2)}_{(1)}(x) &= \bar{\xi}\,\Gamma\, W_c Y^\dagger\, \frac{1}{2} n_- x\, n_+ D_{us}\, h_v \\
J^{(A2)}_{(2)}(x) &= \bar{\xi}\,\Gamma\, W_c Y^\dagger\, \frac{1}{2}\, x_{\perp\mu}\, x_{\perp\nu}\, D^\mu_{us}\, D^\nu_{us}\, h_v \\
J^{(A2)}_{(3)}(x) &= \bar{\xi}\Gamma\, W_c Y^\dagger\, \frac{\not{D}_{us}}{2m_b}\, h_v \\
J^{(A2)}_{(5)}(x) &= -\bar{\xi} i\, \overleftarrow{\not{D}}_{\perp c}\, (i n_+ \overleftarrow{D})^{-1} \frac{\not{n}_+}{2}\, \Gamma\, W_c Y^\dagger\, x_{\perp\mu}\, D^\mu_{us} h_v
\end{aligned} \tag{6.65}
$$

The currents $J^{(B)}$ do not contribute at tree level. Since $W_c = 1$, this discards all the partial derivatives in Eq. (6.23) and Eq. (6.26) acting over W_c. Contribution with and A_c will vanish between the collinear vacuum at tree level and with two A_c will yield a loop correction which is not considered here. Moreover, at tree level in all, but the insertion Lagrangian $\mathcal{L}_{\xi q}$, the collinear covariant derivative can be replaced by simple ∂ derivative. The hard functions are unity. Next, an example of calculation is given.

6.5.1 Sample Calculation at Order λ^2

The hadronic tensor with $J_2^{(A2)\dagger}(x) J^{(A0)}(0)$ looks as:

$$
\begin{aligned}
T^{eff}(q) = \frac{i}{2} \int dx^4 e^{iQ\,x} \\
\langle B(v)| T\left([\bar{h}_v i \overleftarrow{D}^\mu_s i \overleftarrow{D}^\nu_s x_{\perp\nu} x_{\perp\mu} Y \xi^{(0)}](x) [\bar{\xi}^{(0)} Y^\dagger h_v](0)\right) |B(v)\rangle
\end{aligned} \tag{6.66}
$$

Again the soft collinear factorization can be done:

$$
\begin{aligned}
T^{eff}(q) = \frac{i}{2} \int dx^4 e^{iQ\,x} \\
\langle B(v)| T\left([\bar{h}_v i \overleftarrow{D}^{\mu\perp}_s i \overleftarrow{D}^{\nu\perp}_s Y](x_-)_{a\alpha} [Y^\dagger h_v](0)_{b\beta}\right) |B(v)\rangle \times \\
\langle \Omega| T\left([x_{\perp\nu} x_{\perp\mu} \xi^{(0)}](x)_{a\delta} [\bar{\xi}^{(0)}](0)_{b\gamma}\right) |\Omega\rangle \Gamma^\dagger_{\alpha\delta} \Gamma_{\gamma\beta}
\end{aligned}
$$

The soft fields only depend on x_-. Hence the collinear integration

$$
\begin{aligned}
&\int dx_\perp dx_+ e^{i(Q_- x_+)/2} e^{iQ_\perp x_\perp} \langle \Omega| T\left([x_{\perp\nu} x_{\perp\mu} \xi^{(0)}](x)_{a\delta} [\bar{\xi}^{(0)}](0)_{b\gamma}\right) |\Omega\rangle \\
&= i\delta_{ab} \frac{\not{n}_{-\delta\gamma}}{2} \int dl_+\, e^{-il_+ x_-/2} \hat{J}^3_{\nu\mu}(l_+, Q_\perp, Q_-)
\end{aligned} \tag{6.67}
$$

at tree level:

$$
\hat{J}^3_{\nu\mu}(l_+, Q_\perp, Q_-) = \frac{\partial}{\partial Q^\nu_\perp} \hat{J}^\mu_1(l_+, Q_\perp, Q_-) \tag{6.68}
$$

the imaginary part:

$$
J^3_{\nu\mu}(l_+, Q_\perp, Q_-) \equiv \mathrm{Im}\, \hat{J}^3_{\nu\mu}(l_+, Q_\perp, Q_-) \tag{6.69}
$$

With that, the heavy quark operator inside the correlator can be written as:

$$[\bar{h}_v i\overleftarrow{D}_s^{\mu\perp} i\overleftarrow{D}_s^{\nu\perp} Y\bar{\Gamma}](x_-)[Y^\dagger h_v](0) = \int dk_+ e^{-(i/2)k_+x_-}\, F_\perp^{\mu\nu}(k_+) \tag{6.70}$$

with $\bar{\Gamma} = \Gamma^\dagger \frac{n_-}{2}\Gamma$ and

$$F_\perp^{\mu\nu}(k_+) = \bar{h}_v\, iD^{\mu\perp}\, iD^{\nu\perp}\bar{\Gamma}\,\delta(k_+ + iD_+^s)\, h_v = F^{\mu\nu,j}\mathrm{Tr}[P^j\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma] \tag{6.71}$$

Generally, the soft matrix element can be decompose into scalar shape functions:

$$\langle \bar{B}_v| \left[\bar{h}_v i\overleftarrow{D}_s^{\nu} i\overleftarrow{D}_s^{\mu}\right](x_-)_{a\alpha} h_v(0)_{b\beta}|\bar{B}_v\rangle =$$
$$\frac{\delta_{ba}}{N_c}\frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\frac{g_\perp^{\nu\mu}}{2}F_1(x_-) + \frac{\delta_{ba}}{N_c}\left(\frac{-1}{2}\right)\left(\frac{1+\not{v}}{2}\not{n}_-\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}\frac{\epsilon_{\mu\nu}^\perp}{2}F_2(x_-)$$

the Fourier transform of them can be easily identified,ie.:

$$F_1(k_+) = \frac{g_\perp^{\mu\nu}}{2}F^{\mu\nu,1}(k_+) \tag{6.72}$$

The imaginary part of the effective hadronic tensor can be written by:

$$\mathrm{Im}\, T^{eff} = \frac{1}{4}\int dl_+\, J_3\,(m_b - 2E_l + l_+)\, F_1(l_+)\mathrm{Tr}[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma] \tag{6.73}$$

where $J_3 = (g_\perp^{\mu\nu}/2)J_{\nu\mu}^3$ and $Q_- = m_b$. F_2 does not contribute since it is contracted with the symmetric tensor. The hard function H is 1 at tree level and carries the hard scale, J_3 is the jet function, which carries the collinear scale and F_1 is a subleading shape function. Therefore, the factorization formulae beyond leading order hold at $1/m \sim \lambda^2$. Before going further some aspect to remark in the structure of the subleading shape function. The subleading shape function operator contains in comparison with the leading order one two covariant derivative insertions, which will amount up to a correction of $\lambda^4 \sim \Lambda_{QCD}^2/m_b^2$. This is enhanced by the collinear function which two insertions of $x_\perp$ amounting a correction of $\lambda^{-2} \sim m_b/\Lambda_{QCD}$ to give a correction of λ^2. This kind of sub-subleading shape function and enhanced collinear function are a new feature within this approach and are due to the multipole expansion of the soft fields. Generally, $\mathcal{O}(\lambda^n)$ will give subleading shape functions up to $\mathcal{O}(\lambda^{2n})$ and enhanced collinear jets up to $\mathcal{O}(\lambda^{-n})$. In the next section, a classification of the soft shape function operators that appear up to second order will be presented.

6.5.2 Shape Function Operators

The factorization of the soft and collinear degrees of freedom can be performed at all orders. The collinear part, still, perturbative can be calculated and integrated out. The non-perturbative physics are B meson matrix elements of soft fields and has to be classified.

From $J^{(A2)\dagger}(x)J^{(A0)}(0)$ and $J^{(A1)\dagger}(x)J^{(A1)}(0)$, one obtains:

$$\bar{h}_v h_v, \bar{h}_v D_s^\rho h_v\,,\ \bar{h}_v \overleftarrow{D}_s^{\nu\perp}\overleftarrow{D}_s^{\mu\perp} h_v. \tag{6.74}$$

All terms are bilocal. The derivative in the second term can act to the left or to the right. The fact that the second current is chosen to be at zero makes that the double derivative only acts to the left, this term comes from the multipole expansion and comes with $x_\perp$ which vanishes if chosen at zero. These soft products give rise to the shape Function operators:

$$\begin{aligned} S(k_+) &= \bar{h}_v\, \bar{\Gamma}\delta(k_+ + iD_+^s)\, h_v \\ O_1^\rho(k_+) &= \bar{h}_v \bar{\Gamma}\, i\delta(k_+ + iD_+^s)\, D_s^\rho h_v \\ O_2^\rho(k_+) &= \bar{h}_v\, \bar{\Gamma} iD_s^\rho\, \delta(k_+ + iD_+^s)\, h_v \\ F^{\nu\mu}(k_+) &= \bar{h}_v \bar{\Gamma}\, iD^{\perp\nu}\, iD^{\mu\perp}\, \delta(k_+ + iD_+^s)\, h_v. \end{aligned} \tag{6.75}$$

The first term is the leading order shape function operator, the following two are subleading operators with a λ^2 suppression and have already appeared in the literature. The last one, is a new sub-subleading shape function operator.

From $J^{(A1)\dagger}(x)\mathcal{L}^1(y)J^{(A0)}(0)+$ symmetric:

$$\bar{h}_v n_-^\nu\, i\, gF^s_{\mu\,\nu} h_v\,,\ \bar{h}_v \overleftarrow{D}^{\rho\perp} i\, n_-^\nu g F^s_{\mu\,\nu} h_v. \tag{6.76}$$

These terms are tri-local objects. The derivative only acts to the left for the multiple expansion. The corresponding operators:

$$\begin{aligned} O^4_\mu(s_+,k_+) &= \bar{h}_v\, \bar{\Gamma}\delta(s_+ + iD_+^s)\ i\, n_-^\nu g F^s_{\mu\,\nu}\, \delta(k_+ + iD_+^s)\ h_v \\ K^\rho_{\mu\nu}(s_+,k_+) &= \bar{h}_v\, \bar{\Gamma} iD_s^\rho\, \delta(s_+ + iD_+^s)\ i\, gF^s_{\mu\,\nu}\, \delta(k_+ + iD_+^s)\ h_v. \end{aligned} \tag{6.77}$$

The first terms has a λ^2 suppression and the second one is a new sub-subleading operators with a λ^4 suppression.

From $J^{(A0)\dagger}(x)\mathcal{L}^2(y)J^{(A0)}(0)$:

$$\bar{h}_v n_-^\nu n_+^\mu i\, gF^s_{\mu\,\nu} h_v\,,\ \bar{h}_v i\, gF^s_{\mu\,\nu} h_v\,,\ \bar{h}_v \int d^4y \mathcal{L}^{(1/m)}_{HQET}(y) h_v,\ \bar{h}_v[iD^\rho_{us}, i\, g\, n_-^\nu\, F^s_{\mu\nu}]h_v\,. \tag{6.78}$$

These are tri-local. The operators that come up:

$$\begin{aligned} O^4(s_+,k_+) &= \bar{h}_v \bar{\Gamma}\, \delta(s_+ + iD_+^s)\ i\, n_-^\nu n_+^\mu g F^s_{\mu\,\nu}\, \delta(k_+ + iD_+^s)\ h_v \\ O^4_{\mu\nu}(s_+,k_+) &= \bar{h}_v \bar{\Gamma}\, \delta(s_+ + iD_+^s)\ i\, gF^s_{\mu\,\nu}\, \delta(k_+ + iD_+^s)\ h_v \\ t(s_+) &= \int d^4y T\left(\bar{h}_v \bar{\Gamma}\, \delta(s_+ + iD_+^s) h_v\, \mathcal{L}^{(1/m)}_{HQET}(y)\right) \\ G^\rho_{\mu\,\nu}(s_+,k_+) &= \bar{h}_v \bar{\Gamma}\, \delta(s_+ + iD_+^s)\, [iD_s^\rho, i\, g\, F^s_{\mu\nu}]\, \delta(k_+ + iD_+^s)\, h_v \end{aligned}$$

The last one is a new sub-subleading operator.

Finally, from $J^{(A1)\dagger}(x)\mathcal{L}^1(y)\mathcal{L}^1(y)J^{(A0)}(0)$, one obtains

$$\bar{h}_v\, i\, n_-^\nu g F^s_{\mu\,\nu}\, i\, n_-^\beta g F^s_{\alpha\,\beta} h_v\,,\ \bar{h}_v q\bar{q} h_v, \tag{6.79}$$

which are tetra-local objects and the operators are:

$$
\begin{aligned}
&S_{\mu\,\alpha}(s_+, k_+, Q_+) = \\
&\qquad \bar{h}_v\,\bar{\Gamma}\delta(s_+ + iD^s_+)\, n^\nu_-\; i\, gF^s_{\mu\,\nu}\;\delta(k_+ + iD^s_+)\, n^\beta_- i\, g\, F^s_{\alpha\,\beta}\,\delta(Q_+ + iD^s_+)\, h_v \\
&Q(s_+, k_+, Q_+) = \bar{h}_v\bar{\Gamma}\,\delta(s_+ + iD^s_+)\, q\delta(k_+ + iD^s_+)\bar{q}\delta(Q_+ + iD^s_+)h_v.
\end{aligned}
\tag{6.80}
$$

These are new operators. In previous analyses, tetra-local objects were not found. Both of them are suppressed by a factor of λ^4, as the rest of these terms, they come with enhanced collinear jet functions to amount a total λ^2 suppression. In addition, it is the first time that a four-quark shape function at order $1/m_b$ is found. Its phenomenological relevance, still, has to be analyzed. The hadronic tensor up to $\mathcal{O}(\lambda^2)$ for $Q_\perp \neq 0$ in terms of these Operators is given in Appendix B.2. All this operators are non-perturbative objects which have to be sandwiched between B-meson states. It is possible to check the result at tree level by taking free quark states and the Feynman rules of these operators which are given in Fig. 6.5.2 and comparing with the QCD result.

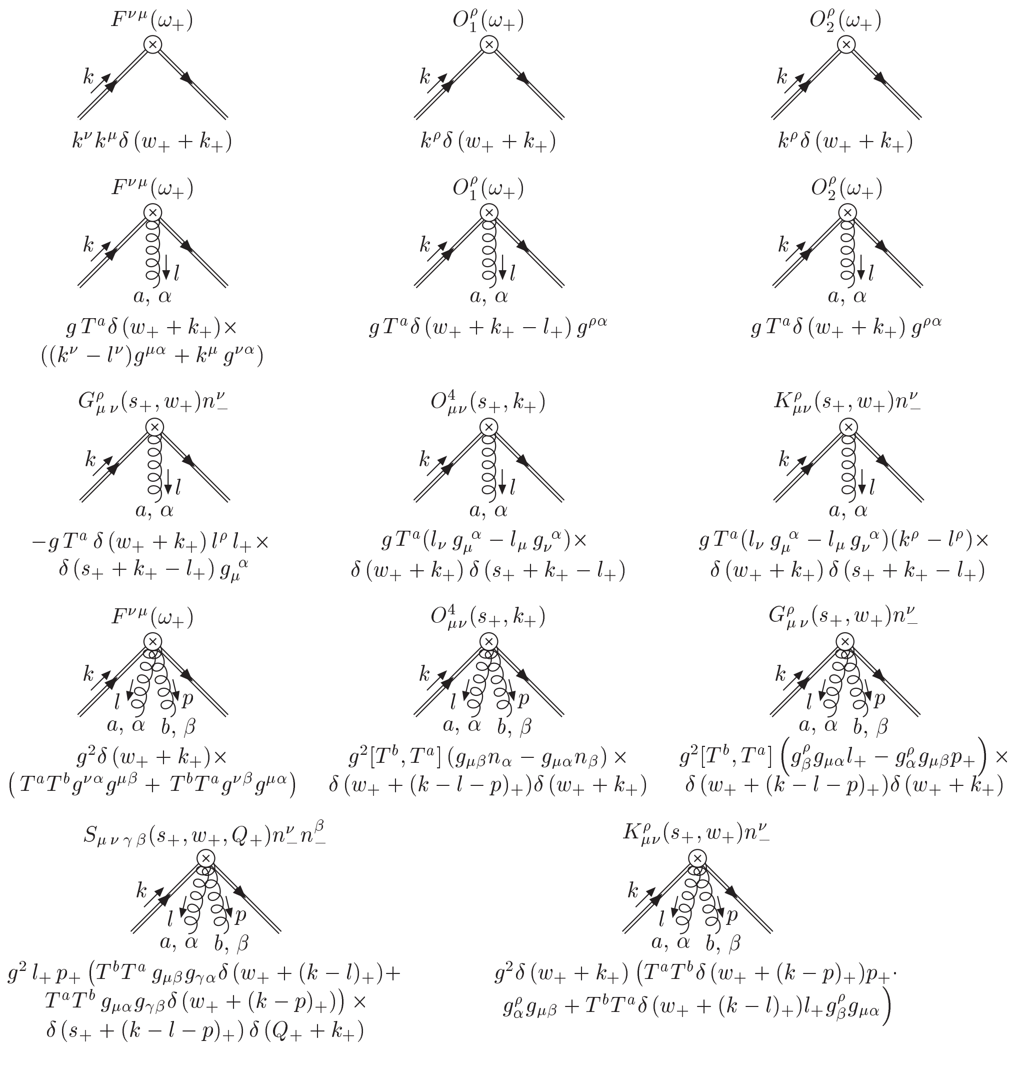

Figure 6.1: Feynam Rules of the Shape Function Operators.

6.5.3 Tree Level QCD

In this section the tree level result of QCD is presented for no gluon, one gluon and two soft gluons emission up to $\mathcal{O}(\lambda^2)$. The b quark is defined to have $p_b = m_b v - k$, with $k \sim \Lambda_{QCD} = m_b\lambda^2$ and in the end point region $Q = q - m_b \cdot v$ with $Q_- \sim m_b$, $Q_+ \sim m_b\lambda^2$, $Q_\perp \sim m_b\lambda$. The emitted gluons carry soft momenta $l \sim p \sim m_b\lambda^2$. At every oder, one can have insertions of the leading order Lagrangian. However, in order to perform the tree level matching from QCD to SCET, the emission of soft components is sufficient to deduce the result.

no-gluon

The hadronic tensor is written from Fig. 6.2:

$$iT = \bar{b}\Gamma\frac{i}{\not{Q}+\not{k}}\Gamma^\dagger b \tag{6.81}$$

Expanding the propagator up to second order:

$$T = \bar{b}\Gamma\left(\frac{-k_\perp^2}{2Q_-(Q_+ + k_+)^2} + \frac{\not{n}_+}{Q_-} + \frac{\not{k}_\perp}{Q_-(Q_+ + k_+)}\right)\Gamma^\dagger b \tag{6.82}$$

which equals the SCET result given in Appendix B.4.1. The first term is reproduced by $J_2^{(A2)}J^{(A0)}$. The second is a local terms and is not reproduced by SCET, since actually the imaginary part of the correlator is matched and this term vanishes, it is not a problem. The third term is reproduced by adding $J_5^{A2}J^{A0}$ and $J_1^{(A1)}J_2^{(A1)}$. The expansion of the b quarks fields is reproduced in all the processes by $J^{(A0)}J_2^{(A3)}$ and its symmetric term.

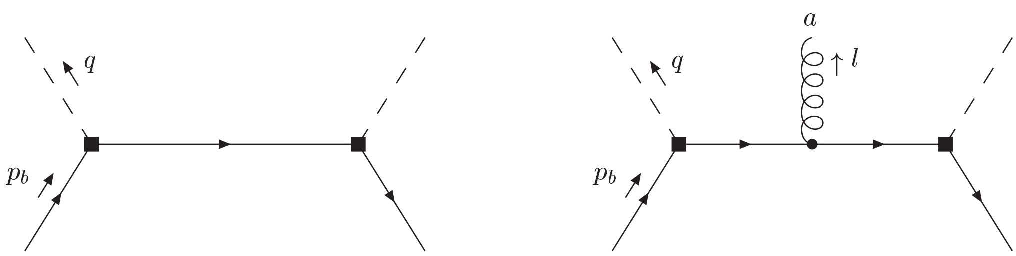

Figure 6.2: no-gluon and one gluon emission Feynman diagrams

one gluon

The QCD digram from Fig. 6.2 for the one gluon case gives:

$$iT = \bar{b}\Gamma\frac{i}{\not{Q}+\not{k}-\not{l}}igT^a\not{\epsilon}^*\frac{i}{\not{Q}+\not{k}}\Gamma^\dagger b. \tag{6.83}$$

Expanding the propagator up to second order, one obtains:

$$T = -\bar{b}\Gamma\left((2k_\perp\cdot\epsilon^\star_\perp - \not{\epsilon}^\star_\perp \not{l}_\perp)\frac{\not{n}_-}{2Q_-}\left(\frac{1}{(Q+k-l)_+}\right)\left(\frac{1}{(Q+k)_+}\right)\right. \\ \left. + \frac{\not{\epsilon}^\star_\perp}{Q_-}\left(\frac{\not{n}_+\not{n}_-}{4}\left(\frac{1}{(Q+k)_+}\right) + \frac{\not{n}_-\not{n}_+}{4}\left(\frac{1}{(Q+k+l)_+}\right)\right)\right)\Gamma^\dagger b \tag{6.84}$$

The first term is reproduced in Appendix B.4.2 by $J_2^{(A2)}J^{(A0)}$ and $J_1^{(A1)}\mathcal{L}_\xi^{(1)}J^{(A0)}$. In addition to these the second terms is obtained from $J^{(A0)}\mathcal{L}_{\xi 2}^{(2)}J^{(A0)}$ $J^{(A0)}\mathcal{L}_{\xi 4}^{(2)}J^{(A0)}$. The last two terms are reproduced by adding $J_2^{(A2)}J^{(A0)}$, $J_1^{(A1)}J_2^{(A1)}$ and $J^{(A0)}\mathcal{L}_\xi^{(1)}J_1^{(A1)}$.

two gluons

The hadronic tensor from Fig. 6.3:

$$iT = \bar{b}\Gamma\frac{i}{\not{Q}+\not{k}-\not{l}-\not{p}}igT^a\not{\epsilon}^\star(p)\frac{i}{\not{Q}+\not{k}-\not{l}}igT^b\not{\epsilon}^\star(l)\frac{i}{\not{Q}+\not{k}}\Gamma^\dagger b \\ + (l\to p, a\to b), \tag{6.85}$$

up to second order is:

$$T = \bar{b}\Gamma T^b T^a\frac{\frac{\not{n}_-}{2}\not{\epsilon}^{b\star}_\perp(p)\not{\epsilon}^{a\star}_\perp(l)}{Q_-}\left(\frac{1}{(Q+k-l-p)_+}\right)\left(\frac{1}{(Q+k)_+}\right)\Gamma^\dagger b \\ + (l\to p, a\to b). \tag{6.86}$$

$J^{(A0)}\mathcal{L}_{\xi 4}^{(2)}J^{(A0)}$, from Appendix B.4.3, antisymmetric in the color indices, reproduces this result and contains and spare term of the form $T^bT^a\epsilon^*\epsilon^*$ which is compensated for the rest of the SCET contributions. The matching from SCET to QCD

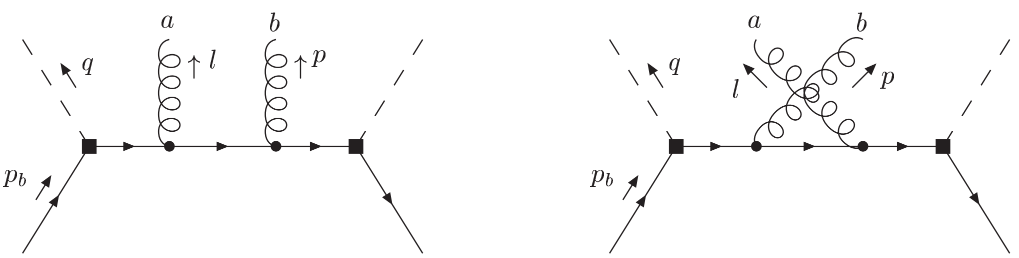

Figure 6.3: two gluon emission Feynman diagrams

is not simple and generally involves the summation of several SCET terms to reproduce some of QCD. As an example, one can study the Abelian case of QCD by taking $T^a = T^b = 1$

$$T(q)_{QCD} = g^2\,\Gamma^\dagger\,\frac{\not{n}_-}{2}\,\Gamma\,\frac{\epsilon^*_\perp\cdot\epsilon^*_\perp}{Q_-} \\ \left(\frac{1}{m-(q-k)_+}\right)\left(\frac{1}{m-(q-k+l+p)_+}\right) \tag{6.87}$$

The result, contrary to the tree digram before expanding, contains only two propagators, and one of them has shrink to a point, becoming local. In SCET, this result is reproduced by adding three contributions, one of them $J_2^{(A1)\dagger}(x)\mathcal{L}_\xi^{(1)}(y)J^{(A0)}(0)$:

$$T(q) = g^2\Gamma^\dagger \frac{\not{n}_-}{2}\,\Gamma\,\frac{\epsilon_\perp\cdot\epsilon_\perp}{Q_-}\left(\frac{1}{m-(q-k)_+}\right)^2 \left(p_+\left(\frac{1}{m-(q-k+p)_+}\right)+l_+\left(\frac{1}{m-(q-k+l)_+}\right)\right) \qquad (6.88)$$

The SCET term contains three propagators. They come from the enhanced jet function, where the $x_\perp$ turn into a derivative, increasing the power of the denominators. This compensates the presence of soft momenta in the numerator which comes from the derivatives in the soft function operators. This momenta can be written in terms of denominators and combined with other SCET terms, where the same manipulation have to be done, yields to the cancellation of propagators reproducing QCD.

In this section, the hadronic tensor up to order λ^2 has been presented. The factorization formulae beyond leading order hold due to the factorization of the soft and collinear degrees of freedom of the leading order Lagrangian. New features have been presented. Up to order λ^2 three convolution variables can appear due to the double insertion of the first subleading Lagrangian.

New enhanced jet functions and extra-suppressed subleading shape functions due to the multiple expansion appear. The latter, of non-perturbative characters are not computable between this approach but are universal and can appear in many decays of B-meson states.

A new feature is the four quark operator, which in addition of the non-perturbative physics of the initial states contains soft fields, non-perturbative physics of the final state. The relevance of this feature has to be analyzed.

Although, non-perturbative object, the tree level matching has been presented in order to check our calculation taking the Feynman rules of our operators.

In the next Section the matrix element between B-meson states of the soft fields that appear in our calculation will be presented. These are parametrized in terms of scalar functions, the shape functions.

6.6 Scalar Shape Functions

In the previous section the shape function operators that appear at tree level have been classified. Leaving the color and Dirac indeces open, taking matrix element between B-meson, in this section, the shape function operators will be decomposed in terms of invariant functions, shape functions. Between two heavy quark states the only possible Dirac structures are 1 and $\gamma^{\mu\top}\gamma_5 = (\gamma^\mu - \not{v}v^\mu)\gamma_5$. The available external vectors are $\hat{n}_-^\mu \equiv n_-^\mu/(n_-v)$ and v^μ and in addition one can use the symmetric metric tensor $g_{\mu\nu}$ and the antisymmetric Levi-cevita tensor which is used in the combination $\epsilon_{\mu\nu}^\perp = i\epsilon_{\mu\nu\rho\sigma}\hat{n}_-^\rho v^\sigma$. The Greek indices refer lorentz(spinor) indices and the Latin to color. Matrix elements have to form a color singlet, therefore, one can define the scalar shape functions:

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} h_v(0)_{b\beta} | \bar{B}_v \rangle = \frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} S(x_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} h_v(0)_{b\beta} \int d^4z \mathcal{L}^{(1/m)}_{HQET}(z) | \bar{B}_v \rangle =$$

$$\frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} S_{1,kin}(x_-) + C_{mag}(mag)(m_b/\mu) S_{1,mag}(x_-)$$

$$\langle \bar{B}_v | \left[\bar{h}_v(-i) \overleftarrow{D}^\mu_s \right] (x_-)_{a\alpha} h_v(0)_{b\beta} | \bar{B}_v \rangle = \frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} [A_1(x_-) v^\mu + A_2(x_-) \hat{n}^\mu_-]$$

$$+ \frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \gamma^{\rho\perp} \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\rho}}{2} A_3(x_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} [iD^\mu_s h_v](0)_{b\beta} | \bar{B}_v \rangle = \frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} [A'_1(x_-) v^\mu + A'_2(x_-) \hat{n}^\mu_-]$$

$$+ \frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \gamma^{\rho\perp} \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\rho}}{2} A'_3(x_-)$$

$$\langle \bar{B}_v | \left[\bar{h}_v i\overleftarrow{D}^\nu_s i\overleftarrow{D}^\mu_s \right] (x_-)_{a\alpha} h_v(0)_{b\beta} | \bar{B}_v \rangle =$$

$$\frac{\delta_{ba}}{N_c} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{g^{\nu\mu}_\perp}{2} B_1(x_-) + \frac{\delta_{ba}}{N_c} \left(\frac{-1}{2} \right) \left(\frac{1+\not{v}}{2} \hat{\not{n}}_- \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\nu}}{2} B_2(x_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} n^\nu_- igF^s_{\mu\perp\nu}(z_-)_{cd} h_v(0)_{b\beta} | \bar{B}_v \rangle =$$

$$\frac{2T^A_{ba} T^A_{cd}}{N_c^2 - 1} \frac{1}{2} \left(\frac{1+\not{v}}{2} \gamma^{\rho\perp} \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\rho}}{2} n_- v C_1(x_-, z_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} n^\mu_+ n^\nu_- igF^s_{\mu\perp\nu}(z_-)_{cd} h_v(0)_{b\beta} | \bar{B}_v \rangle = \frac{2T^A_{ba} T^A_{cd}}{N_c^2 - 1} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} C_2(x_-, z_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} igF^s_{\mu\perp\nu}(z_-)_{cd} h_v(0)_{b\beta} | \bar{B}_v \rangle =$$

$$\frac{2T^A_{ba} T^A_{cd}}{N_c^2 - 1} \left(\frac{-1}{2} \right) \left(\frac{1+\not{v}}{2} \hat{\not{n}}_- \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\nu}}{2} C_3(x_-, z_-)$$

$$\langle \bar{B}_v | \left[\bar{h}_v(-i) \overleftarrow{D}^s_{\rho\perp} \right] (x_-)_{a\alpha} n^\nu_- igF^s_{\mu\perp\nu}(z_-)_{cd} h_v(0)_{b\beta} | \bar{B}_v \rangle =$$

$$\frac{2T^A_{ba} T^A_{cd}}{N_c^2 - 1} \frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{g^\perp_{\mu\rho}}{2} n_- v D_1(x_-, z_-)$$

$$\frac{2T^A_{ba} T^A_{cd}}{N_c^2 - 1} \left(\frac{-1}{2} \right) \left(\frac{1+\not{v}}{2} \hat{\not{n}}_- \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\rho}}{2} D_2(x_-, z_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} \left[in_- D^s, igF^s_{\mu\perp\nu} \right] (z_-)_{cd} h_v(0)_{b\beta} | \bar{B}_v \rangle =$$

$$\frac{2T^A_{ba} T^A_{cd}}{N_c^2 - 1} \left(\frac{-1}{2} \right) \left(\frac{1+\not{v}}{2} \hat{\not{n}}_- \gamma_5 \frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{\epsilon^\perp_{\mu\nu}}{2} D_5(x_-, z_-)$$

$$\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} n^\nu_- igF^s_{\mu\perp\nu}(z_-)_{cd} n^\sigma_- igF^s_{\rho\perp\sigma}(y_-)_{ef} h_v(0)_{b\beta} | \bar{B}_v \rangle =$$

$$\frac{1}{2} \left(\frac{1+\not{v}}{2} \right)_{\beta\alpha} \frac{g^\perp_{\mu\rho}}{2} (n_- v)^2 \left\{ \frac{2\delta_{ba} T^A_{cd} T^A_{ef}}{N_c^2 - 1} E_1(x_-, z_-, y_-) \right.$$

$$
\begin{aligned}
&\qquad -\frac{2if^{ABC}T^A_{cd}T^B_{ef}T^C_{ba}}{N_c(N_c^2-1)}E_2(x_-,z_-,y_-)+\frac{2iN_c d^{ABC}T^A_{cd}T^B_{ef}T^C_{ba}}{(N_c-4)(N_c^2-1)}E_3(x_-,z_-,y_-)\Bigg\}\\
&\left(\frac{-1}{2}\right)\left(\frac{1+\not{v}}{2}\not{n}_-\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}\frac{\epsilon^{\perp}_{\mu\rho}}{2}n_-v\left\{\frac{2\delta_{ba}T^A_{cd}T^A_{ef}}{N_c^2-1}E_4(x_-,z_-,y_-)\right.\\
&\qquad -\frac{2if^{ABC}T^A_{cd}T^B_{ef}T^C_{ba}}{N_c(N_c^2-1)}E_5(x_-,z_-,y_-)+\frac{2iN_c d^{ABC}T^A_{cd}T^B_{ef}T^C_{ba}}{(N_c-4)(N_c^2-1)}E_6(x_-,z_-,y_-)\Bigg\}\\
&\langle\bar{B}_v|\bar{h}_v(x_-)_{a\alpha}h_v(0)_{b\beta}\bar{q}_s(z_-)_{c\gamma}q_s(y_-)_{d\delta}|\bar{B}_v\rangle=\\
&\frac{\delta_{ba}\delta_{dc}}{N_c^2}\left\{\frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}[\delta_{\delta\gamma}F_1(x_-,z_-,y_-)+\not{v}_{\delta\gamma}F_2(x_-,z_-,y_-)\right.\\
&+\not{n}_{-\delta\gamma}F_3(x_-,z_-,y_-)+(\not{v}\not{n}_-)_{\delta\gamma}F_4(x_-,z_-,y_-)]\\
&+\frac{1}{2}\left(\frac{1+\not{v}}{2}\gamma^{\perp\mu}\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}[(\gamma_\mu\gamma_5)_{\delta\gamma}F_5(x_-,z_-,y_-)\\
&(\not{v}\gamma_\mu\gamma_5)_{\delta\gamma}F_6(x_-,z_-,y_-)+(\not{n}_-\gamma_\mu\gamma_5)_{\delta\gamma}F_7(x_-,z_-,y_-)\\
&+(\not{v}\not{n}_-\gamma_\mu\gamma_5)_{\delta\gamma}F_8(x_-,z_-,y_-)]\}\\
&\frac{4T^A_{ba}T^A_{dc}}{(N_c^2-1)^2}\left\{\frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}[\delta_{\delta\gamma}F_9(x_-,z_-,y_-)+\not{v}_{\delta\gamma}F_{10}(x_-,z_-,y_-)\right.\\
&+\not{n}_{-\delta\gamma}F_{11}(x_-,z_-,y_-)+(\not{v}\not{n}_-)_{\delta\gamma}F_{12}(x_-,z_-,y_-)]\\
&+\frac{1}{2}\left(\frac{1+\not{v}}{2}\gamma^{\perp\mu}\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}[(\gamma_\mu\gamma_5)_{\delta\gamma}F_{13}(x_-,z_-,y_-)\\
&(\not{v}\gamma_\mu\gamma_5)_{\delta\gamma}F_{14}(x_-,z_-,y_-)+(\not{n}_-\gamma_\mu\gamma_5)_{\delta\gamma}F_{15}(x_-,z_-,y_-)\\
&+(\not{v}\not{n}_-\gamma_\mu\gamma_5)_{\delta\gamma}F_{16}(x_-,z_-,y_-)]\}
\end{aligned}
\tag{6.89}
$$

In total, 32 scalar shape functions appear. The result in terms of these scalar shape functions for a generic case where $Q_\perp \neq 0$ is in Appendix B.3 where a total of 20 scalar shape functions appear. If one takes the limit $Q_\perp = 0$ the number reduces to 17. The fact that these functions are non-perturbative objects complicate the analysis of these processes and an extraction of V_{ub}, at the moment, seems to be difficult in a model independent way. However, constrains given by RPI, simplification by integrating by parts in the hadronic tensor and equation of motions reduce the number of independent shape functions up to 4. Although, this reduction only has been proven at tree level for the non-abelian case and to all orders for the abelian case [70].

6.7 Factorization Theorem at $1/m_Q$

The analysis at subleading order is complicated since 36 shape functions appear already at tree level(more may be needed beyond it). However, the structure of factorization beyond leading order is clear:

- Express the correlators in terms of SCET currents and Lagrangians

- Perform the hard factorization and the soft-collinear decoupling field redefinition and factorize the matrix elements into:

$$\langle B|\text{soft fields}|B\rangle \times \langle\Omega|\text{collinear fields}|\Omega\rangle$$

- Performs the collinear integration.
- Parametrize shape Functions.

This yields the result:

$$\operatorname{Im} T = \frac{1}{m_b}\sum \underbrace{H}_{m_b^2} * \underbrace{J^{(0,-2)}\,(s_+, l_+, k_+)}_{m_b^2\,\lambda^2} * \underbrace{S^{(2,4)}(s_+, l_+, k_+)}_{m_b^2\,\lambda^4}$$

Therefore, the differential decay rates can be written up to order $1/m_b$ by:

$$\begin{aligned} d\Gamma \propto G_F^2 |V_{CKM}^* \cdot V_{CKM}'| \, &[H_i * J_i\,(s_+) * S(s_+) + \\ &\left.\frac{1}{m_b}\sum H^i * J_i^{(0,-2)}\,(s_+, l_+, k_+) * S^{(2,4)}(s_+, l_+, k_+)\right] \end{aligned}$$

where the upper indeces in S and J mean the power suppresed or enhanced of the soft and the jet function in comparision with the leading orden ones, the sum of the indices sums 2 yielding a correction of order λ^2 .

6.7.1 Beyond Tree Level

Beyond tree level a hard convolutions appear along the jet direction:

$$(\bar{Q}\Gamma q)(x) = \sum_k \int d\hat{s}_1 d\hat{s}_2 \ldots d\hat{s}_n \sum_j \hat{C}_{ij}^{(k)}(\hat{s}_1, \hat{s}_2, \ldots \hat{s}_n) J_j^{(k)}(\hat{s}_1, \hat{s}_2, \ldots \hat{s}_n; x) \quad (6.90)$$

Although, this complicates the form of the factorization theorem in terms on convolution between the hard coefficients and the Jet functions the formalism studied here applies and a similar factorization formula appears. Additional shape functions appear coming from structure not considered in the tree level analysis,ie:

$$\begin{aligned} &\langle \bar{B}_v | \bar{h}_v(x_-)_{a\alpha} \left[iD^s_{\rho\perp}, n_-^\nu igF^s_{\mu\perp\nu} \right] (z_-)_{cd} h_v(0)_{b\beta} | \bar{B}_v \rangle = \\ &\quad \frac{2T^A_{ba}T^A_{cd}}{N_c^2-1}\frac{1}{2}\left(\frac{1+\not{v}}{2}\right)_{\beta\alpha}\frac{g^\perp_{\mu\rho}}{2} n_- v D_3(x_-, z_-) \\ &\quad \frac{2T^A_{ba}T^A_{cd}}{N_c^2-1}\left(\frac{-1}{2}\right)\left(\frac{1+\not{v}}{2}\not{n}_-\gamma_5\frac{1+\not{v}}{2}\right)_{\beta\alpha}\frac{\epsilon^\perp_{\mu\rho}}{2} D_4(x_-, z_-) \end{aligned}$$

A complete proof of factorization beyond tree level including radiative corrections and constrains of the shape functions by RPI invariance and equation of motion will be presented in the near future. The goal is to reduce the number of shape functions closed to the number that appear in the literature by using the approach of [63, 67], (BLM), since will allow the extraction of the V_{ub} with a good accuracy including radiative corrections.

6.8 Comparison with Previous Results

In contrast with the two matching steps presented above, QCD $\to$ SCET $\to$ HQET, in the BLM approach [63, 67] a single step is proposed, matching QCD onto HQET directly. This matching can be only performed at tree level, but beyond tree level HQET, alone, cannot absorb the collinear degrees of freedom that appear in this kinematical region. Since the calculation present above is a tree level, it is worthwhile to compare the methods since the number of shape functions at subleading order is much lower at the BLM and, if possible, it is derisible to reduce the basis given in Equation (6.89) to the BLM one.

Imaginary part of the correlator can be matched at leading order onto:

$$\mathrm{Im}\, T(q) = \frac{1}{2m_Q}\int_{-\infty}^{\infty} d\omega \left[C_0(\omega)\langle O_0(\omega)\rangle + C_{5,0}^{\alpha}(\omega)\langle P_{0,\alpha}(\omega)\rangle + \mathcal{O}\left(\frac{\Lambda_{\mathrm{QCD}}}{m_Q}\right)\right], \quad (6.91)$$

where $O_0(\omega)$ and $P_{0,\alpha}(\omega)$ are the two operators required at leading order in the twist expansion:

$$O_0(\omega) = \bar{h}_v \delta(\omega + in\cdot\hat{D})h_v \quad (6.92)$$

and the correspondent odd parity operator:

$$P_{0,\alpha}(\omega) = \bar{h}_v\gamma_\alpha\gamma_5\delta(\omega + in\cdot\hat{D})h_v \quad (6.93)$$

C_i's are Wilson coefficients. These, in principle, are obtained by calculating the QCD correlator between quark states instead of the B-meson states and matching with (6.91). This, can be done since the short distance coefficients do not depend on non-perturbative physics of the external states. From the tree level correlator calculated in (6.84) after doing the expansion of the effective heavy quark fields

$$b_v(x) = \left(1 + \frac{\not{D}}{2m_Q}\right) h_v(x), \quad (6.94)$$

simplifying the Dirac structure

$$\bar{h}_v\,\Gamma\,h_v = \frac{1}{2}\,\mathrm{Tr}\,(\Gamma\,P_v)\,\bar{h}_v\,h_v - \frac{1}{2}\,\mathrm{Tr}\,(\gamma_\mu\gamma_5\,P_v\,\Gamma\,P_v)\,\bar{h}_v\,\gamma^\mu\gamma_5\,h_v\,, \quad (6.95)$$

and matching with (6.91), one gets for the Wilson coefficients:

$$C_0(v,q,\omega) = \frac{\pi}{2}\mathrm{Tr}\left(P_+\Gamma^\dagger\not{n}\Gamma\right)\delta(1 - n\cdot\hat{q} - \omega) \quad (6.96)$$

$$C_{5,0}^{\alpha}(v,q,\omega) = -\frac{\pi}{2}\mathrm{Tr}\left(s^\alpha\Gamma^\dagger\not{n}\Gamma\right)\delta(1 - n\cdot\hat{q} - \omega) \quad (6.97)$$

For a heavy meson flavor decay, the odd contribution cancels since the matrix element of the axial vector current between B-meson states vanishes by parity invariance.

6.8.1 Subleading Order

At subleading order, the BLM basis of shape functions operators:

$$\begin{aligned} O_1^{\mu i}(\omega) &= \bar{h}_v \left\{ iD^\mu, \delta(in\cdot\hat{D}+\omega)\right\} \Gamma^i h_v \\ O_2^{\mu(\omega)i} &= i\bar{h}_v \left[iD^\mu, \delta(in\cdot\hat{D}+\omega)\right] \Gamma^i h_v \\ O_3^{\mu\nu i}(\omega_1,\omega_2) &= \bar{h}_v \delta(in\cdot\hat{D}+\omega_2)\left\{iD_\perp^\mu, iD_\perp^\nu\right\}\delta(in\cdot\hat{D}+\omega_1)\Gamma^i h_v \qquad (6.98) \\ O_4^{\mu\nu i}(\omega_1,\omega_2) &= g\bar{h}_v \delta(in\cdot\hat{D}+\omega_2) G_\perp^{\mu\nu}\delta(in\cdot\hat{D}+\omega_1)\Gamma^i h_v \\ O_T^i(\omega) &= i\int d^4y\,\frac{1}{2\pi}\int dt\, e^{-i\omega t} T\left(\bar{h}_v(0)\Gamma^i h_v(t)\mathcal{L}_1(y)\right) \qquad (6.99) \end{aligned}$$

where $\Gamma^i \equiv (1, \gamma_\alpha\gamma_5)$ according to Equation (6.44) .

At subleading order the nonlocal OPE in (6.101) is

$$\begin{aligned} -2m_Q \operatorname{Im} T(q) &= \sum_i \int d\omega\, C_0^i(v,q,\omega)\langle O_0^i(\omega)\rangle \qquad (6.100) \\ &+\frac{1}{2m_Q}\sum_i \int d\omega_1 d\omega_2\; C_i^{\mu\nu}(v,q,\omega_1,\omega_2)\langle O_{i,\mu\nu}(\omega_1,\omega_2) + \mathcal{O}\left(\frac{\Lambda_{QCD}^2}{m_b^2}\right) \end{aligned}$$

where the i means a sum over all operators including the odd ones and the second line represent all the subleading operators, which at most appear with two convolution variables. The Wilson coefficients at tree level for $B \to X_s\gamma$ can be found in [63,67] and for the $B \to X_u\, l\,\bar{\nu}$ in [67].

It is easy to reproduce this result from the QCD correlator, at tree level the leading contribution is:

$$T(q) = \int \mathrm{d}^4x\, e^{iQ\cdot x}\,\langle B(v)|\,T\left\{\bar{b}_v(x)\,\Gamma^\dagger\,S_q(x,0)\,\Gamma\, b_v(0)\right\}|B(v)\rangle \qquad (6.101)$$

Expressing the first field and the propagator in momentum space and performing the d^4x integration. The propagator reads as:

$$S_q(Q) = \not{Q} + \not{k}\frac{1}{Q_-(Q+k)_+ + Q_+k_- + k^2} \qquad (6.102)$$

where k is the residual momenta of the heavy quark and scale as $k = m_b(\lambda^2,\lambda^2,\lambda^2)$ and Q is chosen to scale as $Q = m_b(\lambda^2, 1, 0)$. Expanding the QCD fields in terms of the effective ones, the correlator at leading order can be written as:

$$T(q) = \int d^4k\tilde{\bar{h}}_v(k)\bar{\Gamma}h_v(0)\frac{1}{(Q+k)_+} \qquad (6.103)$$

the tilde means the Fourier transform of the field. $\bar{\Gamma} = \Gamma^\dagger\frac{n_-}{2}\Gamma$. Now, if one performs the manipulation:

$$\tilde{\bar{h}}_v(k) = \int d^4x e^{ikx} h_v(x) = \int d^4x e^{ikx} e^{x\partial} h_v(0) \qquad (6.104)$$

integrating over x and k. The correlator can be expressed:

$$T(q) = \int dk_+ \bar{h}_v(0)\delta(k_+ + i\partial_+)\bar{\Gamma} h_v(0)\frac{1}{(Q+k)_+} \tag{6.105}$$

performing the usual manipulations and taking the imaginary part reproduces Equation (6.51). In SCET this is reproduced by T^{eff} which only contains the soft and collinear degrees of freedom. At tree level, the hard contribution is unity. One can see that the Wilson coefficients, C, at the BLM corresponds at tree level with the Jet function calculated in SCET. The subleading Wilson coefficients in terms of the BLM operators at tree level can be obtained going beyond leading order in the expansion of the propagator. The O_3 operator is not reproduced in the SCET derivation. This Operator is obtained from the QCD correlator, with no emission of gluons, from the quadratic term of the light quark propagator. The corresponding term of the correlator:

$$T(q) = -\int dk_+ \bar{\tilde{h}}_v(k)\left(\frac{1}{(Q+k)_+ + i\epsilon}\right)^2 k_\perp^2 \bar{\Gamma} h_v(0) \tag{6.106}$$

Using Eq. (6.104) and inserting a delta function to avoid the square in the denominator:

$$\begin{aligned}\mathrm{ImT}(q) &= -\int dk_+ ds_+ \bar{h}_v(0)\delta(k_+ + i\partial_+)\partial_\perp^2\delta(s_+ + i\partial_+)\bar{\Gamma} h_v(0)\times \\ &\qquad \mathrm{Im}\left(\frac{1}{(Q+k)_+ + i\epsilon}\frac{1}{(Q+s)_+ + i\epsilon}\right) \\ &= \int dk_+ ds_+ C_3(s_+,k_+)_{\mu\nu} O_3^{\mu\nu}(k_+,s_+) \end{aligned} \tag{6.107}$$

one can extract $C^3_{\mu\nu}(s_+,k_+)$ which agrees with the literature. However, it can be written in terms of SCET operator directly using Eq. (6.104).

$$T(q) = -\int dk_+ F^{\mu\nu}(k_+)\frac{g_{\mu\nu}}{2}\left(\frac{1}{(Q+k)_+ + i\epsilon}\right)^2 \tag{6.108}$$

where $F^{\mu\nu} = \bar{h}_v(0)\delta(k_+ + i\partial_+)\partial_\perp^2 \bar{h}_v(0)$. However, for the one gluon emission, one will obtain O_3 instead. In fact, if one restores gauge invariant in Eq.(6.13), applies Eq. (6.104) one obtains

$$T(q) = \langle B(v)|h_v(0)\Gamma^\dagger \frac{1}{\not{Q} + i\not{D}}\Gamma h_v(0)|B(v)\rangle. \tag{6.109}$$

Expanding this expression reproduces exactly the BLM result. The BLM seems to be correct at tree level, but the four quark emission which was not consider there. However, the SCET analysis works beyond tree level and new shape function operators appear. By trivial manipulation, at tree level and to all order for an abelian theory the new shape function operators can be written down in terms of the BLM ones when convoluted with the Jet functions, but, still, it is not clear that this holds beyond tree level for the non-abelian case and the basis required may be larger that the one provided by BLM. Research work in this direction has to be carry out. In the next Chapter, the consequences of studying reparametrization invariance in the HQET framework for the BLM set of operators is studied.

Chapter 7

Reparametrization Invariance in the Endpoint Region

Contents

Assuming that the Basis given by the BLM approach is the correct one, leaving apart the four quark state which was not consider in previous analyses, in this chapter the consequences of studying the RPI in the end point region will be discussed. The leading operators are (6.92,6.93):

$$O_0(\omega) = \bar{h}_v \delta(\omega + (in \cdot D)) h_v \tag{7.1}$$
$$P_0^\alpha(\omega) = \bar{h}_v \delta(\omega + (in \cdot D)) \gamma^\alpha \gamma_5 h_v \tag{7.2}$$

Notice that $P_+ = (1 + \not{v})/2$ and $s_\mu = P_+ \gamma_\mu \gamma_5 P_+$ form a basis in the space of (two-component) spinors projected out by P_+. At subleading order from 6.98:

$$\begin{aligned} O_1^\mu(\omega) &= \bar{h}_v \left\{ (iD^\mu), \delta(\omega + (in \cdot D)) \right\} h_v \\ O_2^\mu(\omega) &= i \, \bar{h}_v \left[(iD^\mu), \delta(\omega + (in \cdot D)) \right] h_v \\ O_3^{\mu\nu}(\omega_1, \omega_2) &= \bar{h}_v \delta(\omega_2 + (in \cdot D)) \left\{ iD_\perp^\mu \, , \, iD_\perp^\nu \right\} \delta(\omega_1 + (in \cdot D)) h_v \\ O_4^{\mu\nu}(\omega_1, \omega_2) &= i \, \bar{h}_v \delta(\omega_2 + (in \cdot D)) \left[iD_\perp^\mu \, , \, iD_\perp^\nu \right] \delta(\omega_1 + (in \cdot D)) h_v \end{aligned} \tag{7.3}$$

for the "spin-independent" operators and from (6.98)

$$\begin{aligned} P_1^{\mu\alpha}(\omega) &= \bar{h}_v \left\{ (iD^\mu), \delta(\omega + (in \cdot D)) \right\} \gamma^\alpha \gamma_5 h_v \\ P_2^{\mu\alpha}(\omega) &= i \, \bar{h}_v \left[(iD^\mu), \delta(\omega + (in \cdot D)) \right] \gamma^\alpha \gamma_5 h_v \\ P_3^{\mu\nu\alpha}(\omega_1, \omega_2) &= \bar{h}_v \delta(\omega_2 + (in \cdot D)) \left\{ iD_\perp^\mu \, , \, iD_\perp^\nu \right\} \delta(\omega_1 + (in \cdot D)) \gamma^\alpha \gamma_5 h_v \\ P_4^{\mu\nu\alpha}(\omega_1, \omega_2) &= i \, \bar{h}_v \delta(\omega_2 + (in \cdot D)) \left[iD_\perp^\mu \, , \, iD_\perp^\nu \right] \delta(\omega_1 + (in \cdot D)) \gamma^\alpha \gamma_5 h_v \end{aligned} \tag{7.4}$$

for the "spin-dependent" ones. The (differential) rates are expressed in terms of convolutions of ω-dependent Wilson coefficients with forward matrix elements of these operators [61]

$$\begin{aligned} d\Gamma = & \int d\omega \left(C_0(\omega)\langle O_0(\omega)\rangle + C^{(5)}_{0,\alpha}(\omega)\langle P^{\alpha}_0(\omega)\rangle \right) \\ & + \frac{1}{m_Q}\sum_{i=1,2}\int d\omega \left(C_{i,\mu}(\omega)\langle O^{\mu}_i(\omega)\rangle + C^{(5)}_{i,\mu\alpha}(\omega)\langle P^{\mu\alpha}_i(\omega)\rangle \right) \\ & + \frac{1}{m_Q}\sum_{i=3,4}\int d\omega_1\, d\omega_2\, (C_{i,\mu\nu}(\omega_1,\omega_2)\langle O^{\mu\nu}_i(\omega_1,\omega_2)\rangle \\ & \qquad\qquad + C^{(5)}_{i,\mu\nu\alpha}(\omega_1,\omega_2)\langle P^{\mu\nu\alpha}_i(\omega_1,\omega_2)\rangle \Big) \\ & + \cdots \end{aligned} \tag{7.5}$$

where $\langle..\rangle$ denotes the forward matrix element with b-Hadron states and the ellipses denote terms originating from time-ordered products with higher order terms of the Lagrangian, which are not relevant for the current purpose.

The main result is that the number of unknown functions that appear at $1/m_Q$ for the B's transitions is reduced [77]. To do that first, the variation of the light-like vectors will be given. Next, an invariant operator will be found, which is related a leading order to the leading order shape function and its expansion will connect subleading order operators with the leading one.

7.1 Reparametrization of the Shape Functions

In the following, the implications of reparametrization invariance for the non-local light-cone operators will be discussed. Similar to the case of local operators, reparametrization-invariant relates combinations of operators containing different orders of the $1/m_Q$ expansion. To investigate this, first, it is necessary to compute the variation of the light cone vectors under a reparametrization transformation, which means that v is varied according to (2.29) and q is kept fixed. Expressing the light-cone vectors in terms of q and v one gets

$$n = \frac{1}{v\cdot q}[2(v\cdot q)v - q] \quad \text{and} \quad \bar{n} = \frac{1}{v\cdot q}q \tag{7.6}$$

then, the variation δ_R under reparametrization is

$$\begin{aligned} \delta_R n_\mu &= \frac{\partial n_\mu}{\partial v_\alpha}\Delta_\alpha = 2\Delta_\mu + \bar{n}_\mu(\bar{n}\cdot\Delta) \\ \delta_R \bar{n}_\mu &= \frac{\partial \bar{n}_\mu}{\partial v_\alpha}\Delta_\alpha = -\bar{n}_\mu(\bar{n}\cdot\Delta) \end{aligned} \tag{7.7}$$

Using this, one can study the variation of

$$\hat{O}_0(\omega) = \bar{h}_v \frac{1}{\omega + (in\cdot D)}\Gamma h_v \tag{7.8}$$

which is of order Λ^2_{QCD} since ω is consider to be of order $\mathcal{O}(\Lambda_{QCD})$. The imaginary part (by replacing $\omega \to \omega + i\epsilon$) of this expression is either $O_0(\omega)$ (for $\Gamma = P_+$) or

$P_0^\alpha(\omega)$ (for $\Gamma = s^\alpha = P_+\gamma^\alpha\gamma_5 P_+$). From (2.32) and (7.7) one obtains:

$$\delta_R(in\cdot D) = -m_Q(n\cdot\Delta) + (\bar n\cdot\Delta)(i\bar n\cdot D) + 2(i\Delta\cdot D) \tag{7.9}$$

and thus

$$\begin{aligned}\delta_R\hat O_0(\omega) &= \bar h_v\{\not\Delta\,,\,\Gamma\}\frac{1}{\omega+(in\cdot D)}h_v \\ &+\bar h_v\frac{1}{\omega+(in\cdot D)}[m_Q(n\cdot\Delta)-(\bar n\cdot\Delta)(i\bar n\cdot D)-2(i\Delta\cdot D)]\frac{1}{\omega+(in\cdot D)}\Gamma h_v \\ &+\mathcal{O}[\Lambda_{QCD}^4/m_Q^2]\end{aligned} \tag{7.10}$$

where terms of subleading order in $1/m_Q$ coming e.g., from the variation of the heavy quark fields, have been omitted.

The first term vanishes due to (2.36) and the fact that Γ is either P_+ or $s_\mu = P_+\gamma_\mu\gamma_5 P_+$. The second term contains a piece of order Λ^2_{QCD} (which is of the same order as $O_0(\omega)$ itself) coming from the variation of the covariant derivative, whereas all the others terms in (7.10) are of higher order.

First, the variation of order Λ^2_{QCD} will be discussed. This can be written as

$$\begin{aligned}\delta_R\hat O_0(\omega) &= \bar h_v\frac{1}{\omega+(in\cdot D)}m_Q(n\cdot\Delta)\frac{1}{\omega+(in\cdot D)}\Gamma h_v+\mathcal{O}(\Lambda^3_{QCD}/m_Q) \\ &= -\left(\frac{\partial}{\partial\omega}\hat O_0(\omega)\right)m_Q(n\cdot\Delta)+\mathcal{O}(\Lambda^3_{QCD}/m_Q)\end{aligned} \tag{7.11}$$

which means that the $\mathcal{O}(\Lambda^2_{QCD})$-variation can be absorbed into a shift $\omega\to\omega - m_Q(n\cdot\Delta)$.

In the following, it will be assumed that Δ does not have a light cone component, i.e., it is only considered $\Delta^\perp$ for which one has $(n\cdot\Delta^\perp)=0$. Note that this also implies $(\bar n\cdot\Delta^\perp)=0$ due to (2.29). In this way (7.10) simplifies to

$$\delta_R^\perp\hat O_0(\omega) = \bar h_v\frac{-2}{\omega+(in\cdot D)}(i\Delta^\perp\cdot D)\frac{1}{\omega+(in\cdot D)}\Gamma h_v+\mathcal{O}(\Lambda^4_{QCD}/m_Q^2)\,. \tag{7.12}$$

The aim is to construct a reparametrization invariant, which is equal to $O_0(\omega)$ to leading order. The variation of $O_0(\omega)$ is of order unity as given in (7.12), and a subleading contribution is needed to compensate this variation. To construct this invariant, one first notes that

$$\delta_R^\perp\left((in\cdot D)+\frac{1}{m_Q}(iD^\perp)^2\right)=0 \tag{7.13}$$

which means that

$$\left(\frac{1}{\omega+(in\cdot D)+\frac{1}{m_Q}(iD^\perp)^2}\right) \tag{7.14}$$

is an exact reparametrization invariant.

Furthermore, using reparametrization invariant fields,

$$h_v^P = h_v+\frac{(i\not D)}{2m_Q}h_v+\frac{1}{4m_Q^2}(iD)^2h_v+\cdots\,,\quad \delta_R^\perp h_v^P=0 \tag{7.15}$$

one can construct the reparametrization-invariant quantity

$$\hat{R}_0(\omega) = \bar{h}_v^P \left(\frac{1}{\omega + (in \cdot D) + \frac{1}{m_Q}(iD^\perp)^2} \right) \Gamma h_v^P \tag{7.16}$$

where Γ is again either P_+ or $s_\mu = P_+\gamma_\mu\gamma_5 P_+$.

This formal expression can be expanded to obtain the reparametrization invariant combination of operators appearing in the twist expansion of inclusive rates. Truncating the expansion yields operators for which reparametrization invariance holds to a certain order in the $1/m_Q$ expansion. One gets

$$\begin{aligned}
\hat{R}_0^{(0)}(\omega) &= \bar{h}_v \frac{1}{\omega + (in \cdot D)} \Gamma h_v \qquad (7.17)\\
\hat{R}_0^{(1)}(\omega) &= \bar{h}_v \frac{1}{\omega + (in \cdot D)} \Gamma h_v - \frac{1}{m_Q} \bar{h}_v \frac{1}{\omega + (in \cdot D)} (iD^\perp)^2 \frac{1}{\omega + (in \cdot D)} \Gamma h_v \\
\hat{R}_0^{(2)}(\omega) &= \bar{h}_v \frac{1}{\omega + (in \cdot D)} \Gamma h_v - \frac{1}{m_Q} \bar{h}_v \frac{1}{\omega + (in \cdot D)} (iD^\perp)^2 \frac{1}{\omega + (in \cdot D)} \Gamma h_v \\
&+ \frac{1}{4m_Q^2} \bar{h}_v (i\not{D}^\perp) \frac{1}{\omega + (in \cdot D)} (i\not{D}^\perp) \Gamma h_v \\
&+ \frac{1}{4m_Q^2} \bar{h}_v \left\{ (iD^\perp)^2 , \frac{1}{\omega + (in \cdot D)} \right\} \Gamma h_v \\
&+ \frac{1}{m_Q^2} \bar{h}_v \frac{1}{\omega + (in \cdot D)} (iD^\perp)^2 \frac{1}{\omega + (in \cdot D)} (iD^\perp)^2 \frac{1}{\omega + (in \cdot D)} \Gamma h_v
\end{aligned}$$

where it holds:

$$\delta_R^\perp \hat{R}_0^{(k)} = \mathcal{O}(\Lambda_{QCD}^{k+3}/m_Q^{k+1}) \tag{7.18}$$

For the case $\Gamma = P_+$ one may rewrite $\hat{R}_0^{(1)}$ in terms of the $O_0(\omega)$ and $O_3(\omega_1, \omega_2)$

$$\hat{R}_0^{(1)}(\omega) = \int \frac{d\sigma}{\omega - \sigma} O_0(\sigma) - \frac{1}{2m_Q} \int \frac{d\sigma_1}{\omega - \sigma_1} \frac{d\sigma_2}{\omega - \sigma_2} g_{\mu\nu}^\perp O_3^{\mu\nu}(\sigma_1, \sigma_2) \tag{7.19}$$

and replace $\omega \to \omega + i\epsilon$ in (7.19) to identify

$$\begin{aligned}
R_0^{(1)}(\omega) &= O_0(\omega) + \frac{1}{\pi} \mathrm{Im} \left(\frac{1}{2m_Q} \int \frac{d\sigma_1}{\omega + i\epsilon - \sigma_1} \frac{d\sigma_2}{\omega + i\epsilon - \sigma_2} g_{\mu\nu}^\perp O_3^{\mu\nu}(\sigma_1, \sigma_2) \right) \\
&= O_0(\omega) - \frac{1}{2m_Q} \int d\sigma_1 \, d\sigma_2 \left(\frac{\delta(\omega - \sigma_1) - \delta(\omega - \sigma_2)}{\sigma_1 - \sigma_2} \right) g_{\mu\nu}^\perp O_3^{\mu\nu}(\sigma_1, \sigma_2) \qquad (7.20)
\end{aligned}$$

to be (up to order Λ_{QCD}^4/m_Q^2) the reparametrization-invariant light-cone operator involving the leading order operator $O_0(\omega)$.

Likewise, for $\Gamma = s^\alpha$ one gets

$$\begin{aligned}
Q_0^{\alpha(1)}(\omega) &= P_0^\alpha(\omega) \qquad (7.21)\\
&- \frac{1}{2m_Q} \int d\sigma_1 \, d\sigma_2 \left(\frac{\delta(\omega - \sigma_1) - \delta(\omega - \sigma_2)}{\sigma_1 - \sigma_2} \right) g_{\mu\nu}^\perp P_3^{\mu\nu\alpha}(\sigma_1, \sigma_2)
\end{aligned}$$

for the spin-dependent reparametrization-invariant quantity up to order Λ_{QCD}^4/m_Q^2.

7.2 Subleading Operators

The other operators of subleading order are not related to $O_0(\omega)$ or $P_0^\alpha(\omega)$. In order to investigate the behavior of $O_1^\mu(\omega)$, the covariant derivative is splitted according to

$$iD^\mu = \frac{1}{2}(\bar{n}^\mu - n^\mu)(in \cdot D) + iD_\perp^\mu \tag{7.22}$$

where the equation of motion for the heavy quark has been used, which implies $(in \cdot D) = -(i\bar{n} \cdot D)$ in $O_1^\mu(\omega)$ as well as in $P_1^{\mu\alpha}(\omega)$. In the same way one can consider $\hat{O}_1^\mu(\omega)$ in which the δ function is replaced by $1/(\omega + (in \cdot D))$. According to (7.22) and splitting $\hat{O}_1^\mu(\omega)$ into $\hat{O}_{1,||}^\mu(\omega)$ and $\hat{O}_{1,\perp}^\mu(\omega)$ one gets

$$\begin{aligned} \hat{O}_{1,||}^\mu(\omega) &= (\bar{n}^\mu - n^\mu)\bar{h}_v(in \cdot D)\frac{1}{\omega + (in \cdot D)}\Gamma h_v \\ &= (\bar{n}^\mu - n^\mu)\left[\bar{h}_v \Gamma h_v - \omega \bar{h}_v \frac{1}{\omega + (in \cdot D)}\Gamma h_v\right] \end{aligned} \tag{7.23}$$

Taking the imaginary part (after $\omega \to \omega + i\epsilon$) for $\Gamma = P_+$ results in

$$O_{1,||}^\mu(\omega) = (n^\mu - \bar{n}^\mu)\omega \bar{h}_v \delta(\omega + (in \cdot D))h_v = (n^\mu - \bar{n}^\mu)\,\omega\, O_0(\omega) \tag{7.24}$$

which means that $O_{1,||}^\mu(\omega)$ is completely given in terms of $O_0(\omega)$. The same arguments apply for the spin-dependent operator $P_1^{\mu\alpha}(\omega)$, where $P_{1||}^{\mu\alpha}(\omega)$ is entirely given in terms of $P_0^\alpha(\omega)$

However, for the perpendicular pieces $O_{1,\perp}^\mu(\omega)$ and $P_{1\perp}^{\mu\alpha}(\omega)$ the reparametrization variation is

$$\delta_R^\perp O_{1,\perp}^\mu(\omega) = -2m_Q \Delta_\perp^\mu O_0(\omega) + \mathcal{O}(\Lambda_{QCD}^4/m_Q) \tag{7.25}$$

$$\delta_R^\perp P_{1,\perp}^{\mu\alpha}(\omega) = -2m_Q \Delta_\perp^\mu P_0^\alpha(\omega) + \mathcal{O}(\Lambda_{QCD}^4/m_Q) \tag{7.26}$$

which means that this variation contains a contribution of the same order as the operator itself, which would need to be compensated for by some other subleading operator. However, there is no such an operator, and so, one concludes that reparametrization invariance requires that only $O_{1,||}^\mu(\omega)$ and $P_{1||}^{\mu\alpha}(\omega)$ contribute to a physical quantity.

Using the same arguments for the case of $O_2^\mu(\omega)$ and $P_2^{\mu\alpha}(\omega)$, one gets $O_{2||}^\mu(\omega) = 0 = P_{2||}^{\mu\alpha}(\omega)$. However, unlike for $O_1^\mu(\omega)$ and $P_1^{\mu\alpha}(\omega)$, a reparametrization transformation yields

$$\delta_R^\perp O_{2,\perp}^\mu(\omega) = \mathcal{O}(\Lambda_{QCD}^4/m_Q) \tag{7.27}$$

$$\delta_R^\perp P_{2,\perp}^{\mu\alpha}(\omega) = \mathcal{O}(\Lambda_{QCD}^4/m_Q) \tag{7.28}$$

since these operators involve a commutator rather than an anticommutator, and hence, they will in general contribute.

Finally, nothing new can be obtained for $O_4^{\mu\nu}$ or $P_4^{\mu\nu\alpha}$ from reparametrization invariance; these operators are related through reparametrization to higher order terms, which have not yet been classified.

7.3 Applications

One immediate consequence of the above result concerns the matching coefficients for light cone operators. Since physical observables such as (differential) rates are reparametrization invariants, the matching coefficients $C_0(\omega)$ of $O_0(\omega)$ and the one of $O_3^{\mu\nu}(\omega_1, \omega_2)$ have to be related, such that

$$\begin{aligned} d\Gamma &= \int d\omega \left(C_0(\omega)\langle R_0(\omega)\rangle + D_{0\alpha}(\omega)\langle P_0^\alpha(\omega)\rangle \right) \qquad (7.29) \\ &= \int d\omega \left(C_0(\omega)\langle O_0(\omega)\rangle + D_{0\alpha}(\omega)\langle P_0^\alpha(\omega)\rangle \right) \\ &- \frac{1}{2m_Q}\int d\omega C_0(\omega) \int d\sigma_1\, d\sigma_2 \left(\frac{\delta(\omega-\sigma_1)-\delta(\omega-\sigma_2)}{\sigma_1-\sigma_2} \right) g_{\mu\nu}^\perp \langle O_3^{\mu\nu}(\sigma_1,\sigma_2)\rangle \\ &- \frac{1}{2m_Q}\int d\omega D_{0\alpha}(\omega) \int d\sigma_1\, d\sigma_2 \left(\frac{\delta(\omega-\sigma_1)-\delta(\omega-\sigma_2)}{\sigma_1-\sigma_2} \right) g_{\mu\nu}^\perp \langle P_3^{\mu\nu\alpha}(\sigma_1,\sigma_2)\rangle \end{aligned}$$

which can be compared to (7.5), yielding to the reparametrization-invariance relation between the coefficients

$$C_{3,\mu\nu}(\sigma_1,\sigma_2) = -\frac{1}{2}\int d\omega C_0(\omega) \left(\frac{\delta(\omega-\sigma_1)-\delta(\omega-\sigma_2)}{\sigma_1-\sigma_2} \right) g_{\mu\nu}^\perp \qquad (7.30)$$

$$C_{3,\mu\nu\alpha}^{(5)}(\sigma_1,\sigma_2) = -\frac{1}{2}\int d\omega C_{0,\alpha}^{(5)}(\omega) \left(\frac{\delta(\omega-\sigma_1)-\delta(\omega-\sigma_2)}{\sigma_1-\sigma_2} \right) g_{\mu\nu}^\perp \qquad (7.31)$$

Relation (7.30) has been shown at tree level by explicit calculation for the case of $B \to X_s\gamma$ in [61] and holds also for the case $B \to X_u \ell\bar{\nu}_\ell$ [60], but here, it is stated that such a relation is a consequence of reparametrization invariance and thus, has to hold including radiative corrections.

In particular, it must hold for the renormalization kernel of the subleading operators $O_3^{\mu\nu}(\omega_1,\omega_2)$ and $P_3^{\mu\nu\alpha}(\omega_1,\omega_2)$. While up to now, only the renormalization kernel of the leading order term has been investigated [138] [139], a relation like (7.30) has to relate the kernel of $O_3^{\mu\nu}(\omega_1,\omega_2)$ to the one of $O_0(\omega)$, and the kernel of $P_3^{\mu\nu\alpha}(\omega_1,\omega_2)$ will be related to the one of $P_0^\alpha(\omega)$.

In this way, reparametrization invariance reduces the number of unknown functions, parametrizing e.g. the photon spectrum of $B \to X_s\gamma$ to subleading order. Following [63], the non-vanishing matrix elements leading to independent functions are

$$\begin{aligned} \langle B(v)|O_0(\omega)|B(v)\rangle &= 2m_B f(\omega) \qquad (7.32) \\ \langle B(v)|O_3^{\mu\nu}(\omega_1,\omega_2)|B(v)\rangle &= 2m_B\, g_2(\omega_1,\omega_2) g_\perp^{\mu\nu} \\ \langle B(v)|P_{2,\alpha}^{\mu}(\omega)|B(v)\rangle &= 2m_B\, h_1(\omega)\varepsilon_{\perp,\alpha}^{\mu} \\ \langle B(v)|P_{4,\alpha}^{\mu\nu}(\omega_1,\omega_2)|B(v)\rangle &= 2m_B\, h_2(\omega_1,\omega_2)\varepsilon_{\rho\sigma\alpha\beta}\, g_\perp^{\mu\rho} g_\perp^{\nu\sigma} v^\beta \\ \langle B(v)|O_T(\omega)|B(v)\rangle &= 2m_B\, t(\omega)\,, \end{aligned}$$

where $\varepsilon_\perp^{\mu\nu}$ is

$$\varepsilon_\perp^{\mu\nu} = \varepsilon^{\mu\nu\alpha\beta} v_\alpha n_\beta\,, \qquad (7.33)$$

and $\varepsilon^{0123} = 1$. Furthermore, $O_T(\omega)$ is the contribution originating from the time-ordered product of the leading-order operator $O_0(\omega)$ with the $1/m_Q$ corrections to the Lagrangian; the precise definition can be found in [63].

The contributions of g_2 and h_2 can be gathered into a function of a single variable

$$G_2(\sigma) = \int d\omega_1\, d\omega_2\, g_2(\omega_1, \omega_2) \left[\frac{\delta(\sigma - \omega_1) - \delta(\sigma - \omega_2)}{\omega_1 - \omega_2}\right] \tag{7.34}$$

$$H_2(\sigma) = \int d\omega_1\, d\omega_2\, h_2(\omega_1, \omega_2) \left[\frac{\delta(\sigma - \omega_1) - \delta(\sigma - \omega_2)}{\omega_1 - \omega_2}\right] \tag{7.35}$$

which is, at least for g_2, not surprising, since it is a consequence of reparametrization invariance. In [63], the conclusion was reached that the four universal functions $F(\omega) = f(\omega) + t(\omega)/(2m_Q)$, $G_2(\omega)$, $h_1(\omega)$ and $H_2(\omega)$ are needed to parametrize the subleading twist contributions to heavy-to-light decays.

From reparametrization invariance, one concludes that the functions $F(\omega)$ and $G_2(\omega)$ have to appear always in the same combination, such that

$$\mathcal{F}(\omega) = f(\omega) + \frac{1}{2m_Q} t(\omega) - \frac{1}{m_Q} G_2(\omega) \tag{7.36}$$

is a single universal function. This has been confirmed at tree level by explicit calculation, though this should hold to all orders in $\alpha_s(m_Q)$.

Chapter 8

Conclusions

Heavy flavour physics, and in particular the physics of the bottom quark, has become an excellent scenario for testing some parameters of the SM. Big efforts have been made both experimentally and theoretically for the extraction of the V_{ub} and V_{cb} parameters of the CKM matrix.

The theoretical progress in this area is related with the establishment of two effective theories HQET and SCET. These two, together with Heavy Quark Expansion, settle the basis to study and control the QCD interaction in heavy to light weak currents transitions.

In this thesis a short introduction of these theories has been presented in the first two chapters in order to prepare the stage to consider several aspect of QCD heavy to light weak currents. In the first place, heavy to light currents in the limit in which the momentum of the light quark is small in comparison with the mass of the heavy quark has been studied in detail in the Chapter 4 by means of HQET and Heavy Quark Expansions. A complete next-to-leading analysis has been presented including $1/m$ corrections for a general Dirac structure. Based on a renormalons analysis the behaviour of the leading order Wilson coefficients and ratios of B and B^* mesons decays constants have been calculated assuming that the cancellation of the renormalons ambiguity beyond the large-β_0 limit holds. Our main result is the asymptotic behaviour of the perturbative series of the measurable quantity f_{B^*}/f_B. The large two-loop correction in this ratio was observed in [50]; here a model-independent result for higher orders which continue this trend has been presented. The smallest contribution seems to be the 3-loop one, and it is about 4% (though this small value is due to a partial cancellation between the leading order and the next-to-leading one, and thus is not quite reliable). Therefore, calculation of this 3-loop correction is meaningful (and it is actually possible, using the technique of [132]), while the 4-loop calculation seems to be no necessary. The main results are:

1. All Dirac structures for heavy to light currents can be reduced to the study of two different cases; two currents with Spin 0 and two with Spin 1.

2. The coefficients B_2^1 and $B_2^{\not{v}}$ of the subleading operator O_2 for the currents with $\Gamma = 1$ and $\not{v}$ are related by (4.61). One-loop results for the generic Γ (4.27) and for the tensor current (4.91) are also new and were presented in [52].

3. The heavy-quark symmetry relations [111] for B- and B^*-meson matrix elements of subleading operators get non-trivial radiative corrections (4.97), (4.102).

4. An explicit renormalization group equation at the next to leading order for the B and B^* meson decay constants have been presented [52].

5. By an explicit calculation the cancellation of the IR renormalon of the leading matching coefficient against the UV ones of the subleading operators has been shown.

6. Behaviour of the Borel images of perturbative series near the leading singularity $u = \frac{1}{2}$ for the matching coefficients (5.25), (5.35) at the NLO, at the NNLO for $\hat{f}^T_{B^*}/f_{B^*}$ (5.39), and f_{B^*}/f_B (5.43) and at the NNNLO for $m/\hat{m}$ (5.30), have been found. The powers of $\frac{1}{2} - u$ are exact; further corrections are suppressed by positive integer powers of $\frac{1}{2} - u$.

7. The normalization factors $N_{0,1,2}$ cannot be found within this approach; they are some unknown numbers of order unity. Comparison with the exact 3-loop result from Appendix A suggests that the normalization factor N_0 is smaller that its large-β_0 value 1, namely, $N_0 \sim 0.27$. This conclusion is in a qualitative agreement with the estimate (5.33), especially if the problematic 3-loop correction in it is omitted.

8. Logarithmics branching is a new feature of this problem; it follows from the fact that the anomalous dimensions matrices cannot be diagonalized.

9. Asymptotics of perturbative coefficients c_L at $L \gg 1$ for the matching coefficients (5.27), (5.36), and for the same ratios (5.31), (5.40), (5.44) have been found. The powers of $n = L - 1$ are exact; further corrections are suppressed by positive integer powers of $1/n$. Logarithmic terms follow from the same property of the anomalous dimensions.

Second, motivated by experimental constrains, the spectra of inclusive decays for B meson has been studied in the end point region of the spectrum by a two step matching up to second order in the parameter of the expansion. At leading order, it was shown that the spectra can be written in terms of a convolution of a hard, a jet, and a soft function. Here, it has been proven that the same pattern beyond leading order holds. The fact that the subleading Lagrangians couple soft and collinear degrees of freedom does not spoil the factorization theorem. Since the leading order Lagrangian does not couple them, at every order of the expansion the decoupling can be done at the cost of introducing in the soft function information of the subleading collinear Lagrangians, obtaining at every order more complicated shape functions. Up to order λ^2, 32 scalar shape functions appear at tree level. The hadronic tensor has been calculated up to order λ^2 at tree level in terms of shape function operators Appendix B.2. Up to 20 shape functions appear when written in terms of scalar shape functions Appendix B.3. However, some of them may be related by equation of motions and reparametrization invariance constrains.

The tree level result has been checked taking Feynman rules of the shape functions operators and comparing with QCD, Appendix B.4. The formalism present here is quite general and can be applied systematically to include subleading order terms and include radiative corrections. The main results are:

1. The factorization formulae hold beyond leading order
2. In comparison with previous results up to three convolution variables are needed to proof factorization up to λ^2. This is due to the insertion of two subleading Lagrangians. In general at order λ^n it will necessary $2n+1$ integration variables.
3. New shape functions appear. In particular at order λ^2 appear subleading shape functions suppress by a factor of λ^4. This is due to the multipole expansion of the soft fields.
4. At order λ^2 enhanced jet functions appear up to order λ^{-2} accompanying the extra suppress shape functions. This is due to the multipole expansion.
5. Four quark state subleading shape functions at order $1/m$ appear. Their phenomenological relevance has to be analyzed.

These results have been compared with previous analysis of subleading shape functions valid, only, at tree level done at the BLM. A tree level, when taking Feynman rules of the shape function operators both results can give the right answer since match to QCD. The basis of shape functions required by BLM is lower, and therefore it is desirable to reduce the set of functions close to the BLM one, with the exception of the four quark state which has not been considered there. This can be done at tree level and at all orders for an abelian theory [70], but for the real QCD is need it more reseach.

Assuming that the set of shape functions present here collapses to the BLM one. The consequences of considering reparametrization invariance for the subleading shape functions have been studied within the HQET framework. Looking at the first subleading terms, reparametrization relates the leading order shape function to one of the subleading matrix elements, leading to identical matching coefficients for the two contributions. As a practical consequence, the spectra of inclusive heavy-to-light transitions are parametrized in terms of three unknown universal functions, once the first subleading terms are included. This last conclusion has to be revised in the SCET framework.

The appearance of the universal functions opens the possibility of measuring V_{ub} in a model independent way including subleading terms. The idea is to measure the universal shape function from the experiment in a process like $B \to X_s \gamma$ and use this information in $B \to X_u \bar{\nu} l_\nu$ to obtain the V_{ub} parameter. Research work in this direction touches the phenomenological applications of SCET, which are the topic of ongoing studies.

Appendix A

The Perturbative Series for $m/\hat{m}$

The ratio of the on-shell mass m and the renormalization-group invariant $\overline{\mathrm{MS}}$ mass $\hat{m}$ is the series

$$\frac{m}{\hat{m}} = 1 + \sum_{L=1}^{\infty} c_L \left(\frac{\alpha_s(\mu_0)}{4\pi}\right)^L$$

in the n_l-flavour $\alpha_s(\mu_0)$ ($\mu_0 = e^{-5/6}m$). Using the 3-loop relation [132,140] [1] between m and $m(m)$ (omitting the m_c effect known at two loops [140]) together with the 4-loop β-function [142] and the mass anomalous dimension [143,144], as well as [145, 146]

$$\alpha_s(m) = \alpha_s'(m)\left[1 + \frac{1}{9}\left(39C_F - 32C_A\right)T_F\left(\frac{\alpha_s'(m)}{4\pi}\right)^2 + \cdots\right],$$

we obtain

$$\begin{aligned}
c_1 &= C_F\left[\tfrac{3}{2} - \left(\tfrac{15}{2}C_F + 8C_A\right)\frac{1}{\beta_0'} + 3\left(11C_F + 7C_A\right)\frac{C_A}{\beta_0'^2}\right],\\
c_2 &= C_F\Bigg\{\left(\pi^2 + \tfrac{3}{4}\right)\beta_0' + \left(-6\zeta_3 - 8\pi^2\log 2 + 5\pi^2 + \tfrac{121}{8}\right)C_F\\
&\quad + \left(12\zeta_3 + 4\pi^2\log 2 - 5\pi^2 + \tfrac{149}{16}\right)C_A + \left(4\pi^2 - \tfrac{46}{3}\right)T_F\\
&\quad + \Big[+\tfrac{69}{2}C_F^2 + \left(66\zeta_3 - \tfrac{169}{4}\right)C_FC_A + \left(-66\zeta_3 + \tfrac{131}{12}\right)C_A^2\\
&\qquad + \tfrac{50}{3}C_FT_F + \tfrac{160}{9}C_AT_F\Big]\frac{1}{\beta_0'}\\
&\quad + \tfrac{1}{48}\Big(1350C_F^3 + 504CF^2C_A - 3948C_FC_A^2 - 483C_A^3\\
&\qquad - 3520C_FC_AT_F - 2240C_A^2T_F\Big)\frac{1}{\beta_0'^2}\\
&\quad + \tfrac{3}{2}\left(-15C_F^2 - 5C_FC_A + 7C_A^2\right)\left(11C_F + 7C_A\right)\frac{C_A}{\beta_0'^3}\\
&\quad + \tfrac{9}{2}C_F\left(11C_F + 7C_A\right)^2\frac{C_A^2}{\beta_0'^4}\Bigg\},
\end{aligned}$$

[1] The three-loop coefficient in it had been found numerically [141] before the analytical result [132] was obtained.

$$
\begin{aligned}
c_3 = C_F \Big\{ & \left(12\zeta_3 + \pi^2 + \tfrac{3}{4}\right) \beta_0'^2 \\
& + \Big[\Big(128a_4 + 100\zeta_3 + \tfrac{16}{3}\log^4 2 + \tfrac{32}{3}\pi^2\log^2 2 - 32\pi^2\log 2 \\
& \qquad - \tfrac{22}{9}\pi^4 + \tfrac{47}{2}\pi^2 + \tfrac{383}{6}\Big) C_F \\
& \quad + \Big(-64a_4 - 61\zeta_3 - \tfrac{8}{3}\log^4 2 - \tfrac{16}{3}\pi^2\log^2 2 + 16\pi^2\log 2 \\
& \qquad + \tfrac{2}{9}\pi^4 + \tfrac{13}{3}\pi^2 + \tfrac{1513}{48}\Big) C_A \\
& \quad + \Big(48\zeta_3 - \tfrac{4}{9}\pi^2 - \tfrac{22}{3}\Big) T_F\Big] \beta_0' \\
& + \Big(768a_4 + 80\zeta_5 + 4\pi^2\zeta_3 + 305\zeta_3 + 32\log^4 2 - 32\pi^2\log^2 2 \\
& \quad - 508\pi^2\log 2 + \tfrac{4}{3}\pi^4 + \tfrac{673}{3}\pi^2 + \tfrac{7595}{48}\Big) C_F^2 \\
& + \Big(-384a_4 - 200\zeta_5 + 76\pi^2\zeta_3 + 10\zeta_3 - 16\log^4 2 + 16\pi^2\log^2 2 \\
& \quad + \tfrac{794}{3}\pi^2\log 2 - \tfrac{2}{3}\pi^4 - \tfrac{1315}{6}\pi^2 - \tfrac{73579}{288}\Big) C_F C_A \\
& + \Big(30\zeta_5 - 51\pi^2\zeta_3 - 150\zeta_3 - \tfrac{16}{3}\pi^2\log 2 + \tfrac{5}{2}\pi^4 + \tfrac{139}{6}\pi^2 + \tfrac{431}{8}\Big) C_A^2 \\
& + \Big(\tfrac{944}{3}\zeta_3 + \tfrac{64}{3}\pi^2\log^2 2 - \tfrac{896}{9}\pi^2\log 2 - \tfrac{56}{9}\pi^4 + \tfrac{2602}{27}\pi^2 - \tfrac{1073}{9}\Big) C_F T_F \\
& + \Big(40\zeta_5 - 8\pi^2\zeta_3 - \tfrac{424}{3}\zeta_3 - \tfrac{32}{3}\pi^2\log^2 2 - \tfrac{1856}{9}\pi^2\log 2 \\
& \quad + \tfrac{28}{9}\pi^4 + \tfrac{4972}{27}\pi^2 - \tfrac{1829}{9}\Big) C_A T_F \\
& + \left(-\tfrac{64}{15}\pi^2 + \tfrac{1640}{27}\right) T_F^2 + 4(6\zeta_3 + 7) C_{FF} \\
& + \Big[\Big(-67\zeta_3 + 60\pi^2\log 2 - \tfrac{75}{2}\pi^2 - \tfrac{7589}{48}\Big) C_F^3 \\
& \quad + \Big(-440\zeta_5 + 111\zeta_3 + 34\pi^2\log 2 + \tfrac{61}{2}\pi^2 + \tfrac{110269}{288}\Big) C_F^2 C_A \\
& \quad + \Big(220\zeta_5 - 201\zeta_3 - 32\pi^2\log 2 + 61\pi^2 + \tfrac{4175}{12}\Big) C_F C_A^2 \\
& \quad + \Big(220\zeta_5 + 175\zeta_3 + \tfrac{27583}{432}\Big) C_A^3 \\
& \quad - \left(30\pi^2 + \tfrac{40}{3}\right) C_F^2 T_F + \Big(-\tfrac{880}{3}\zeta_3 - 32\pi^2 + \tfrac{7673}{18}\Big) C_F C_A T_F \\
& \quad + \left(\tfrac{880}{3}\zeta_3 + \tfrac{1258}{27}\right) C_A^2 T_F - \tfrac{1000}{27} C_F T_F^2 - \tfrac{3200}{81} C_A T_F^2 \\
& \quad + 264(\zeta_3 - 1) C_{FF} C_A - 16(21\zeta_3 - 2) C_{FA}\Big] \frac{1}{\beta_0'} \\
& + \Big[-\tfrac{4815}{16} C_F^4 + \Big(-693\zeta_3 - 264\pi^2\log 2 + 165\pi^2 + \tfrac{2955}{4}\Big) C_F^3 C_A \\
& \quad + \Big(1205\zeta_3 - 36\pi^2\log 2 - 60\pi^2 + \tfrac{179561}{432}\Big) C_F^2 C_A^2 \\
& \quad + \Big(428\zeta_3 + 84\pi^2\log 2 - 105\pi^2 - \tfrac{49189}{96}\Big) C_F C_A^3 + \Big(-616\zeta_3 + \tfrac{763}{8}\Big) C_A^4 \\
& \quad - 125 C_F^3 T_F + \left(132\pi^2 - \tfrac{1988}{3}\right) C_F^2 C_A T_F + \Big(84\pi^2 - \tfrac{3769}{9}\Big) C_F C_A^2 T_F
\end{aligned}
$$

$$
\begin{aligned}
&- \tfrac{3689}{18} C_A^3 T_F + \tfrac{4400}{27} C_F C_A T_F^2 + \tfrac{2800}{27} C_A^2 T_F^2 + \left(-\tfrac{3872}{3}\zeta_3 + \tfrac{5324}{9} \right) C_{FF} C_A^2 \\
&+ \left(\tfrac{4576}{3}\zeta_3 - \tfrac{1408}{9} \right) C_{FA} C_A + \left(-\tfrac{704}{3}\zeta_3 + \tfrac{80}{9} \right) C_{AA} \Big] \frac{1}{\beta_0'^2} \\
+ \Big[&- \tfrac{1125}{16} C_F^5 + 1656 C_F^4 C_A + \left(2178\zeta_3 + \tfrac{785}{8} \right) C_F^3 C_A^2 \\
&+ \left(-792\zeta_3 - \tfrac{132323}{96} \right) C_F^2 C_A^3 + \left(-1386\zeta_3 - \tfrac{2219}{4} \right) C_F C_A^4 + \tfrac{147}{4} C_A^5 \\
&+ 1100 C_F^3 C_A T_F + \tfrac{3200}{3} C_F^2 C_A^2 T_F - 280 C_F C_A^3 T_F - \tfrac{980}{3} C_A^4 T_F \Big] \frac{1}{\beta_0'^3} \\
+ \tfrac{1}{16} \Big(&1350 C_F^4 - 2664 C_F^3 C_A - 6140 C_F^2 C_A^2 + 637 C_F C_A^3 + 784 C_A^4 \\
&- 3520 C_F^2 C_A T_F - 2240 C_F C_A^2 T_F \Big) (11 C_F + 7 C_A) \frac{C_A}{\beta_0'^4} \\
+ \tfrac{9}{4} &C_F \left(-15 C_F^2 + 6 C_F C_A + 14 C_A^2 \right) (11 C_F + 7 C_A)^2 \frac{C_A^2}{\beta_0'^5} \\
+ \tfrac{9}{2} &C_F^2 (11 C_F + 7 C_A)^3 \frac{C_A^3}{\beta_0'^6} ,
\end{aligned}
$$

where $a_4 = \mathrm{Li}_4\left(\frac{1}{2}\right)$, and

$$
C_{FF} = \frac{d_F^{abcd} d_F^{abcd}}{T_F^2 N_A} , \quad C_{FA} = \frac{d_F^{abcd} d_A^{abcd}}{T_F N_A} , \quad C_{AA} = \frac{d_A^{abcd} d_A^{abcd}}{N_A}
$$

(see notations in [142, 144]). For $SU(N_c)$ with $T_F = \frac{1}{2}$,

$$
C_{FF} = \frac{N_c^4 - 6 N_c^2 + 18}{24 N_c^2} , \quad C_{FA} = \frac{N_c (N_c^2 + 6)}{24} , \quad C_{AA} = \frac{N_c^2 (N_c^2 + 36)}{24} .
$$

Appendix B

Hadronic Tensor up to $\mathcal{O}(\lambda^2)$

B.1 Tree Level Jet Functions

In this section, the required set of Jet Functions is presented:

$$
\begin{aligned}
J_0(Q_+, Q_\perp) &= \frac{1}{Q_+ + Q_\perp^2/Q_-} \\
J_1^\mu(Q_+, Q_\perp) &= -\frac{\partial}{\partial Q_{\perp\mu}} J_0(Q_+, Q_\perp) \\
J_2^\mu(Q_+, Q_\perp) &= \frac{Q_\perp^\mu}{Q_-} J_0(Q_+, Q_\perp) \\
J_3^{\nu\mu}(Q_+, Q_\perp) &= \frac{\partial}{\partial Q_{\perp\nu}} J_1^\mu(Q_+, Q_\perp) \\
J_4^{\nu\mu}(Q_+, Q_\perp) &= \frac{\partial}{\partial Q_{\perp\nu}} J_2^\mu(Q_+, Q_\perp) \\
J_5^{\nu\mu}(Q_+^1, Q_+^2, Q_\perp) &= \frac{\partial}{\partial Q_{\perp\nu}} J_0(Q_+^1, Q_\perp) J_1^\mu(Q_+^2, Q_\perp)
\end{aligned} \tag{B.1}
$$

B.2 Shape Function operators

The correlator in terms of the shape functions operators is presented:

B.2.1 Order(λ)

1. For $J_{(1)}^{(A1)\dagger}(x) J_{(0)}^{(A0)}$

$$T(q) = \frac{1}{2} \int dk_+ \, O_2^\mu(k_+) \, J_{1\mu}((Q-s)_+) \times \mathrm{Tr}\left[P_+ \Gamma^\dagger \frac{\not{n}_-}{2} \Gamma \right] \tag{B.2}$$

2. $J_{(2)}^{A1\dagger}(x) J_0^{A0}$

$$T(vq) = \frac{1}{2} \int dk_+ \, S(k_+) \, J_2^\mu(Q_+ - k_+) \times \mathrm{Tr}\left[P_+ \Gamma^\dagger \frac{\not{n}_+}{2} \gamma_{\perp\mu} \frac{\not{n}_-}{2} \Gamma \right] \tag{B.3}$$

3. $J^{A1\dagger}_{(2)}(x)J^{A0}_0$

$$T(q) = \frac{1}{2}\int dk_+\, S(k_+)\, J_2^\mu((Q-k)_+) \times \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_{\perp\mu}\frac{\not{n}_+}{2}\Gamma\right] \tag{B.4}$$

4. $J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(1)}_\xi(y)J^{A0}_0$

$$T(q) = \frac{1}{2}\int dk_+\, ds_+ O^4_{\mu\nu}(k_+)n_-^\nu$$
$$J_0((Q-k)_+)J_1^\mu((Q-s)_+) \times \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{B.5}$$

B.2.2 Order(λ^2)

$J^{A2\dagger}_{(1)}(x)J^{A0}_{(0)}(0)$

$$T(vq) = \frac{-1}{2}\int ds_+\, O_2^\mu(s_+)n_{+\mu}J_1^\rho((Q-s)_+)\frac{Q_{\perp\rho}}{Q_-}\mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{B.6}$$

$J^{A2\dagger}_{(2)}(x)J^{A0}_{(0)}(0)$

$$T(vq) = \frac{1}{4}\int ds_+\, F_\perp^{\nu\mu}(s_+)J^3_{\nu\mu}((Q-s)_+)\mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{B.7}$$

$J^{A2\dagger}_{(3)}(x)J^{A0}_{(0)}(0)$

$$T(q) = \frac{1}{2}\int ds_+\, \frac{O_2^\rho(s_+)}{2m_b}\, J_0((Q-k)_+) \times \mathrm{Tr}\left[P_+\gamma_\rho\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\right] \tag{B.8}$$

$J^{(A0)\dagger}_{(0)}(x)J^{A2}_{(3)}(0)$

$$T(vq) = \frac{-1}{2}\int ds_+\, \frac{O_1^\rho(s_+)}{2m_b}\, J_0((Q-k)_+) \times \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\gamma_\rho\right] \tag{B.9}$$

$J^{A2\dagger}_{(5)}(x)J^{A0}_{(0)}(0)$

$$T(q) = \frac{1}{2}\int ds_+\, O_2^\mu(s_+)\, J^4_{\mu\nu}((Q-k)_+) \times \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_+}{2}\gamma_\perp^\nu\frac{\not{n}_-}{2}\Gamma\right] \tag{B.10}$$

$J^{A1\dagger}_{(1)}(x)J^{A1}_{(2)}(0)$

$$T(q) = \frac{1}{2}\int ds_+\, O_2^\mu(s_+)\, J^4_{\mu\nu}((Q-k)_+) \times \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_\perp^\nu\frac{\not{n}_+}{2}\Gamma\right] \tag{B.11}$$

$J^{A1\dagger}_{(2)}(x)J^{A1}_{(2)}(0)$

$$T(vq) = \frac{1}{2}\int ds_+\, S(s_+)\, J_2^\rho((Q-k)_+)\frac{Q_\perp^\mu}{Q_-} \times \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_+}{2}\gamma_{\perp\rho}\gamma_{\perp\mu}\Gamma\right] \tag{B.12}$$

$J^{A1\dagger}_{(1)}(x)\mathcal{L}^{(1)}_\xi(y)J^{A0}_0$

$$T(q) = \frac{1}{2}\int ds_+\, dk_+ K^\alpha_{\mu\nu}(s_+,k_+)\; n_-^\nu\; J^\mu_{5\alpha}((Q-s)_+,(Q-k)_+,Q_\perp)$$
$$Tr\left[P_+\,\Gamma^\dagger\,\frac{\not{n}_-}{2}\,\frac{\not{n}_+}{2}\,\frac{\not{n}_-}{2}\,\Gamma\right] \tag{B.13}$$

$J^{A1\dagger}_{(2)}(x)\mathcal{L}^{(1)}_\xi(y)J^{A0}_0$

$$T(q) = \frac{-1}{2}\int ds_+ dk_+\, O^4_{\mu\nu}(s_+,k_+) n_-^\nu\, J_2^\rho(Q-s)_+)J_1^\mu((Q-k)_+)$$
$$\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_+}{2}\gamma_{\perp\rho}\frac{\not{n}_-}{2}\Gamma\right] \tag{B.14}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(1)}_\xi(y)J^{A1}_{(2)}$

$$T(q) = \frac{1}{2}\int ds_+ dk_+\, O^4_{\mu\nu}(s_+,k_+)J_0((Q-k)_+)J_4^{\rho\mu}((Q-k)_+)$$
$$\times\,\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\rho}\frac{\not{n}_+}{2}\Gamma\right] \tag{B.15}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(1)}_\xi(y)\mathcal{L}^{(1)}_\xi(z)J^{A0}_{(0)}$

$$T(vq) = \frac{1}{4}\int ds_+ dk_+ dl_+\; S^{acdefb}_{\mu\,\nu\,\alpha\,\beta}(s_+,k_+,l_+) n_-^\nu n_-^\beta\,\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right]\delta_{ac}\delta_{de}\delta_{fb}$$
$$J_0(Q-s)_+)J_5^{\mu\alpha}((Q-k)_+,(Q-l)_+) \tag{B.16}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 1}(y)J^{A0}_{(0)}$

$$T(q) = \frac{-1}{4}\int ds_+ dk_+\, O^4_{\mu\nu}(s_+,k_+)n_+^\mu n_+^\nu J_{2\rho}((Q-s)_+)J_1^\rho((Q-k)_+)$$
$$\times\,\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \tag{B.17}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 2}(y)J^{A0}_{(0)}$

$$T(q) = \frac{-1}{4}\int ds_+\, dk_+\, G^{\rho}_{\mu\nu}(s_+,k_+)n^{\nu}_-\, J_0((Q-s)_+)J^{\mu}_{3\rho}((Q-k)_+)$$
$$\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \tag{B.18}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 3}(y)J^{A0}_{(0)}$

$$T(q) = \frac{1}{4}\int ds_+ dk_+\, O^4_{\mu\nu}(s_+,k_+)J^{\mu}_2((Q-s)_+)J^{\rho}_1((Q-k)_+)$$
$$\times\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\rho}\gamma^{\nu}_{\perp}\Gamma\right] \tag{B.19}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 4}(y)J^{A0}_{(0)}$

$$T(vq) = \frac{-1}{4}\int ds_+ dk_+\, O^4_{\mu\nu}(s_+,k_+)J_0((Q-s)_+)J^{\mu\rho}_4((Q-k)_4)$$
$$\times\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\gamma^{\nu}_{\perp}\gamma_{\perp\rho}\frac{\not{n}_+}{2}\frac{\not{n}_-}{2}\Gamma\right] \tag{B.20}$$

B.3 Scalar Shape Functions

$J^{A2\dagger}_{(1)}(x)J^{A0}_{(0)}(0)$

$$T(q) = \frac{-1}{2}\int ds_+\, (A_1(s_+)v^{\mu} + A_2(s_+)\hat{n}^{\mu}_-)n_{+\mu}J^{\rho}_1((Q-s)_+)\frac{Q_{\perp\rho}}{Q_-}$$
$$\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \tag{B.21}$$

$J^{A2\dagger}_{(2)}(x)J^{A0}_{(0)}(0)$

$$T(q) = \frac{1}{4}\int ds_+ B_1(s_+)\frac{g^{\nu\mu}_{\perp}}{2}J^3_{\mu\nu}((q-s)_+)\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \tag{B.22}$$

$J^{A2\dagger}_{(3)}(x)J^{A0}_{(0)}(0)$

$$T(vq) = \frac{1}{4m_b}\int ds_+\, \Bigg((A_1(s_+)v^{\rho} + A_2(s_+)\hat{n}^{\rho}_-)\mathrm{Tr}\left[P_+\gamma_\rho\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right]$$
$$+A_3(s_+)\frac{\epsilon^{\perp}_{\rho\mu}}{2}\mathrm{Tr}\left[s^{\mu}_{\perp}\gamma_\rho\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right]\Bigg)\, J_0((Q-s)_+) \tag{B.23}$$

$J^{(A0)\dagger}_{(0)}(x) J^{A2}_{(3)}(0)$

$$T(vq) = \frac{-1}{4m_b} \int ds_+ \left((A'_1(s_+)v^\rho + A'_2(s_+)\hat{n}^\rho_-) \mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\gamma_\rho\right] + A'_3(s_+)\frac{\epsilon^\perp_{\rho\mu}}{2}\mathrm{Tr}\left[s^\mu_\perp \Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\gamma_\rho\right]\right) J_0((Q-s)_+) \tag{B.24}$$

$J^{A2\dagger}_{(5)}(x) J^{A0}_{(0)}(0)$

$$T(vq) = \frac{1}{2}\int ds_+ A_3(s_+) J^4_{\nu\mu}((Q-s)_+)\frac{\epsilon^\perp_{\mu\rho}}{2}\mathrm{Tr}\left[s^\rho_\perp \Gamma^\dagger \frac{\not{n}_+}{2}\gamma^\nu_\perp \frac{\not{n}_-}{2}\Gamma\right] \tag{B.25}$$

$J^{A1\dagger}_{(1)}(x) J^{A1}_{(2)}(0)$

$$T(vq) = \frac{1}{2}\int ds_+ A_3(s_+)\frac{\epsilon^\perp_{\mu\rho}}{2} J^4_{\nu\mu}((Q-s)_+)\mathrm{Tr}\left[s^\rho_\perp \Gamma^\dagger \frac{\not{n}_-}{2}\gamma^\nu_\perp \frac{\not{n}_+}{2}\Gamma\right] \tag{B.26}$$

$J^{A1\dagger}_{(2)}(x) J^{A1}_{(2)}(0)$

$$T(vq) = \frac{1}{2}\int ds_+\, S(s_+)\, J^\rho_1((Q-k)_+)\frac{Q^\mu_\perp}{Q_-}\mathrm{Tr}\left[P_+\Gamma^\dagger \frac{\not{n}_+}{2}\gamma_{\perp\rho}\gamma_{\perp\mu}\Gamma\right] \tag{B.27}$$

$J^{A1\dagger}_{(1)}(x)\mathcal{L}^{(1)}_\xi(y) J^{A0}_0$

$$T(q) = \frac{1}{2}\int ds_+\, dk_+ n_- v D_1(s_+,k_+)\frac{g^\perp_{\mu\alpha}}{2} J^{\mu\alpha}_5((Q-s)_+,(Q-k)_+)\ \mathrm{Tr}\left[P_+\,\Gamma^\dagger\,\frac{\not{n}_-}{2}\,\Gamma\right] \tag{B.28}$$

$J^{A1\dagger}_{(2)}(x)\mathcal{L}^{(1)}_\xi(y) J^{A0}_0$

$$T(vq) = \frac{-1}{2}\int ds_+ dk_+\frac{\epsilon^\perp_{\mu\sigma}}{2} n_- v C_1(s_+,k_+) J^\rho_2((Q-s)_+) J^\mu_1((Q-k)_+)\ \mathrm{Tr}\left[s^\sigma_\perp \Gamma^\dagger \frac{\not{n}_+}{2}\gamma_{\perp\rho}\frac{\not{n}_-}{2}\Gamma\right] \tag{B.29}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(1)}_\xi(y) J^{A1}_{(2)}$

$$T(q) = \frac{1}{2}\int ds_+ dk_+\frac{\epsilon^\perp_{\mu\sigma}}{2} n_- v C_1(s_+,k_+) J_0((Q-s)_+) J^{\mu\rho}_4((Q-k)_+)\ \mathrm{Tr}\left[s^\sigma_\perp \Gamma^\dagger \frac{\not{n}_-}{2}\gamma_{\perp\rho}\frac{\not{n}_+}{2}\Gamma\right] \tag{B.30}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(1)}_{\xi}(y)\mathcal{L}^{(1)}_{\xi}(z)J^{A0}_{(0)}$

$$T(vq) = \frac{1}{4}\int ds_+dk_+dl_+ \left(E_1(s_+,k_+,l_+) + \frac{E_2(s_+,k_+,l_+)}{2} + \frac{E_3(s_+,k_+,l_+)}{2}\right)\times \mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right]\frac{g^{\perp}_{\mu\alpha}}{2}(n_-v)^2 J_0((Q-s)_+)J_5^{\mu\alpha}((Q-k)_+,(Q-l)_+) \tag{B.31}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 1}(y)J^{A0}_{(0)}$

$$T(vq) = \frac{-1}{4}\int ds_+dk_+\, C_2(s_+,k_+)J_{2\mu}((Q-s)_+)J_1^{\mu}((Q-k)_+)\,\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \tag{B.32}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 2}(y)J^{A0}_{(0)}$

$$T(q) = \frac{-1}{4}\int ds_+\, dk_+\, \frac{g^{\perp}_{\rho\mu}}{2}n_-vD_3(s_+,k_+)J_0((Q-s)_+)J^{\mu}_{3\rho}((Q-k)_+)\,\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right] \tag{B.33}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 3}(y)J^{A0}_{(0)}$

$$T(q) = \frac{-1}{4}\int ds_+dk_+\, \frac{\epsilon^{\perp}_{\mu\nu}}{2}C_3(s_+,k_+)J_2^{\mu}((Q-s)_+)J_1^{\rho}((Q-k)_+)\,\mathrm{Tr}\left[s^{\perp}_{\sigma}\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\rho}\gamma_\perp^{\nu}\Gamma\right]n_-^{\sigma} \tag{B.34}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(2)}_{\xi 4}(y)J^{A0}_{(0)}$

$$T(q) = \frac{1}{4}\int ds_+dk_+\, \frac{\epsilon^{\perp}_{\mu\nu}}{2}C_3(s_+,k_+)J_0((Q-s)_+)J_4^{\rho\mu}((Q-k)_+)\,\mathrm{Tr}\left[s^{\perp}_{\sigma}\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^{\nu}\gamma_{\perp\rho}\Gamma\right]n_-^{\sigma} \tag{B.35}$$

$J^{A0\dagger}_{(0)}(x)\mathcal{L}^{(1)}_{\xi q1}(y)\mathcal{L}^{(1)}_{\xi q2}(z)J^{A0}_{(0)}$

$$T(vq) = \frac{-1}{4}\int ds_+dk_+dl_+ \left(\frac{N^2-1}{2N^2}\left(F_2(s_+,k_+,l_+)\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^{\alpha}\not{v}\gamma^{\perp}_{\alpha}\frac{\not{n}_-}{2}\Gamma\right]+ +F_5(s_+,k_+,l_+)\mathrm{Tr}\left[s_\perp^{\mu}\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^{\alpha}\gamma_\mu\gamma_5\gamma^{\perp}_{\alpha}\frac{\not{n}_-}{2}\Gamma\right]+\right.\right.$$

$$F_6(s_+,k_+,l_+)\mathrm{Tr}\left[s_\perp^\mu\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^\alpha\not{v}\gamma_\mu\gamma_5\gamma_\alpha^\perp\frac{\not{n}_-}{2}\Gamma\right]\Big)+$$
$$\frac{-1}{N(N^2-1)}\Big(F_{10}(s_+,k_+,l_+)\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^\alpha\not{v}\gamma_\alpha^\perp\frac{\not{n}_-}{2}\Gamma\right]+$$
$$+F_{13}(s_+,k_+,l_+)\mathrm{Tr}\left[s_\perp^\mu\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^\alpha\gamma_\mu\gamma_5\gamma_\alpha^\perp\not{n}_-2\Gamma\right]+$$
$$F_{14}(s_+,k_+,l_+)\mathrm{Tr}\left[s_\perp^\mu\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^\alpha\not{v}\gamma_\mu\gamma_5\gamma_\alpha^\perp\frac{\not{n}_-}{2}\Gamma\right]\Big)\Big)$$
$$J_0((Q-s)_+)J_0((Q-k)_+)J_0((Q-(s+k+l))_+) \tag{B.36}$$

B.4 Tensor Current at Tree Level

B.4.1 No Gluons Emission

$J_1^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q)=\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\,k_-\,J_1^\rho((Q+k)_+)\frac{Q_{\perp\rho}}{Q_-} \tag{B.37}$$

$J_2^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q)=\frac{1}{2}\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma k^\mu k^\nu J^1_{\mu\nu}((Q+k)_+) \tag{B.38}$$

$J_3^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q)=\frac{-k^\rho}{2m_b}\gamma_\rho\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\,J_0((Q+k)_+) \tag{B.39}$$

$J^{(A0)\dagger}(x)J_3^{(A2)}(0)$

$$T(q)=\frac{k^\rho}{2m_b}\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma J_0((Q+k)_+) \tag{B.40}$$

$J_5^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(vq)=-\Gamma^\dagger\frac{\not{n}_+}{2}\gamma_\perp^\nu\frac{\not{n}_-}{2}\Gamma k^\mu J^4_{\mu\nu}((Q+k)_+) \tag{B.41}$$

$J_1^{(A1)\dagger}(x)J_2^{(A1)}(0)$

$$T(q)=-\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\nu}\frac{\not{n}_+}{2}\Gamma k^\mu J^4_{\mu\nu}((Q+k)_+) \tag{B.42}$$

$J_2^{(A1)\dagger}(x)J_2^{(A1)}(0)$

$$T(q) = \Gamma^\dagger \gamma_{\perp\rho}\gamma_{\perp\mu}\frac{\not{n}_+}{2}\Gamma J_2^\rho((Q+k)_+)\frac{Q_\perp^\mu}{Q_-} \tag{B.43}$$

B.4.2 One Gluon Emission

$J_1^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = -gT^a\epsilon_-^\star\,\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma J_1^\rho((Q+k)_+)\frac{Q_{\perp\rho}}{Q_-} \tag{B.44}$$

$J_2^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = gT^a\frac{1}{2}\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma(2k-l)^\nu\epsilon^{\star\mu}J_{\nu\mu}^3((Q+k)_+) \tag{B.45}$$

$J_3^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = gT^a\frac{\epsilon^{\star\rho}}{2m_b}\gamma_\rho\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma J_0((Q+k)_+) \tag{B.46}$$

$J^{(A0)\dagger}(x)J_3^{(A2)}(0)$

$$T(vq) = -gT^a\frac{\epsilon^{\star\rho}}{2m_b}\,\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\gamma_\rho J_0((Q+k-l)_+) \tag{B.47}$$

$J_5^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = gT^a\epsilon^{\star\mu}\Gamma^\dagger\frac{\not{n}_+}{2}\gamma_\perp^\nu\frac{\not{n}_-}{2}\Gamma J_{\mu\nu}^4((Q+k)_+) \tag{B.48}$$

$J_1^{(A1)\dagger}(x)J_2^{(A1)}(0)$

$$T(vq) = gT^a\epsilon^{\star\mu}\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\nu}\frac{\not{n}_+}{2}\Gamma J_{\mu\nu}^4((Q+k)_+) \tag{B.49}$$

$J_1^{(A1)\dagger}(x)\mathcal{L}_\xi^{(1)}(y)J_0^{(A0)}$

$$T(q) = \frac{1}{2}\,l_+\epsilon_\mu^*(k-l)^\alpha Tr\left[P_+\Gamma^\dagger\,\frac{\not{n}_-}{2}\,\Gamma\right]J_{5\alpha}^\mu((Q+k-l)_+,(Q+k)_+) \tag{B.50}$$

$J_2^{(A1)\dagger}(x)\mathcal{L}_\xi^{(1)}(y)J_0^{(A0)}$

$$T(q) = -\epsilon_\mu^\star l_+\Gamma^\dagger\frac{\not{n}_+}{2}\gamma_{\perp\rho}\frac{\not{n}_-}{2}\Gamma J_2^\rho((Q+k-l)_+)J_1^\mu((Q+k)_+) \tag{B.51}$$

$J_0^{(A0)\dagger}(x)\mathcal{L}_\xi^{(1)}(y)J_2^{(A1)}$

$$T(q) = gT^a l_+ \epsilon_\mu^\star \Gamma^\dagger \frac{\not{n}_-}{2}\gamma_{\perp\rho}\frac{\not{n}_+}{2}\Gamma J_0((Q+k)_+)J_4^{\mu\rho}((Q+k-l)_+) \quad \text{(B.52)}$$

$J_0^{(A0)\dagger}(x)\mathcal{L}_{\xi 2}^{(2)}(y)J_0^{(A0)}$

$$T(q) = \frac{1}{2}l_+ l^\rho \epsilon_\mu^\star \Gamma^\dagger \frac{\not{n}_-}{2}\Gamma J_0((Q+k-l)_+)J_{3\rho}^\mu((Q+k)_+) \quad \text{(B.53)}$$

$J_0^{(A0)\dagger}(x)\mathcal{L}_{\xi 3}^{(2)}(y)J_0^{(A0)}$

$$T(q) = \frac{1}{2}(l_\nu \epsilon_\mu^\star - l_\mu \epsilon_\nu^\star)\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_{\perp\rho}\gamma_\perp^\nu \Gamma J_2^\mu((Q+k-l)_+)J_1^\rho((Q+k)_+) \quad \text{(B.54)}$$

$J_{(0)}^{A0\dagger}(x)\mathcal{L}_{\xi 4}^{(2)}(y)J_{(0)}^{A0}$

$$T(q) = \frac{-1}{2}(l_\nu \epsilon_\mu^\star - l_\mu \epsilon_\nu^\star)\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_\perp^\nu \gamma_{\perp\rho}\Gamma J_0((Q+k-l)_+)J_4^{\rho\mu}((Q+k)_+) \quad \text{(B.55)}$$

B.4.3 Two Gluon Emission

$J_2^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = \frac{1}{2}g^2\{T^a, T^b\}\,\mathrm{Tr}\epsilon^{\star\mu}\epsilon^{\star\nu}\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma J_{\mu\nu}^3((Q+k)_+) \quad \text{(B.56)}$$

$J_{(1)}^{A1\dagger}(x)\mathcal{L}_\xi^{(1)}(y)J_0^{A0}$

$$\begin{aligned} T(q) = \frac{1}{2}\,Tr\left[P_+\,\Gamma^\dagger\,\frac{\not{n}_-}{2}\,\Gamma\right]\,\epsilon_a^{*\alpha}(l)\epsilon_{b\mu}^*(l)T^aT^b p_+ \\ J_{5\alpha}^\mu((Q+k-p)_+,(Q+k)_+) + (a\to b, p\to l) \end{aligned} \quad \text{(B.57)}$$

$J_{(0)}^{A0\dagger}(x)\mathcal{L}_\xi^{(1)}(y)\mathcal{L}_\xi^{(1)}(z)J_{(0)}^{A0}$

$$\begin{aligned} T(q) = \frac{1}{2}\mathrm{Tr}\left[P_+\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\right]\,\epsilon_a^{*\alpha}(l)\epsilon_{b\mu}^*(l)T^aT^b p_+ l_+ J_0((Q+k-l-p)_+) \\ J_{5\alpha}^\mu((Q+k-l)_+,(Q+k)_+) + (a\to b, p\to l) \end{aligned} \quad \text{(B.58)}$$

$J_{(0)}^{A0\dagger}(x)\mathcal{L}_{\xi 2}^{(2)}(y)J_{(0)}^{A0}$

$$\begin{aligned} T(q) = \frac{-1}{2}\left[T^b, T^a\right]\epsilon_\mu^\star\epsilon^{\star\rho}(l_+ - p_+)\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma \\ J_0((Q+k-l-p)_+)J_{3\rho}^\mu((Q+k)_+) \end{aligned} \quad \text{(B.59)}$$

$J_{(0)}^{A0\dagger}(x)\mathcal{L}_{\xi 3}^{(2)}(y)J_{(0)}^{A0}$

$$T(vq) = \frac{1}{2}\left[T^b, T^a\right](\epsilon_\mu^{\star b}\epsilon_\nu^{\star a} - \epsilon_\mu^{\star a}\epsilon_\nu^{\star b})\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\rho}\gamma_\perp^\nu\Gamma \\ J_2^\mu((Q+k-l)_+)J_1^\rho((Q+k)_+) \tag{B.60}$$

$J_{(0)}^{A0\dagger}(x)\mathcal{L}_{\xi 4}^{(2)}(y)J_{(0)}^{A0}$

$$T(q) = \frac{-1}{2}(\epsilon_\mu^{\star b}\epsilon_\nu^{\star a} - \epsilon_\mu^{\star a}\epsilon_\nu^{\star b})\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_\perp^\nu\gamma_{\perp\rho}\Gamma\left[T^b, T^a\right] \\ J_0((Q+k-l-p)_+)J_4^{\rho\mu}((Q+k)_+) \tag{B.61}$$

B.5 Radiative Decay, $Q_\perp = 0$

B.5.1 No Gluons

$J_2^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = \frac{1}{2}\frac{k_\perp^2}{Q_-}\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right)^2 \tag{B.62}$$

$J_3^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = \frac{-1}{2}\frac{k^\rho}{2m_b}\gamma_\rho\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right) \tag{B.63}$$

$J^{(A0)\dagger}(x)J_3^{(A2)}(0)$

$$T(q) = \frac{1}{2}\frac{k^\rho}{2m_b}\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\gamma_\rho\left(\frac{1}{m_b-(q-k)_+}\right) \tag{B.64}$$

$J_5^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = \frac{-1}{2}\frac{k_\perp^\rho}{m_b}\Gamma^\dagger\frac{\not{n}_+}{2}\gamma_{\perp\rho}\frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right) \tag{B.65}$$

$J_1^{(A1)\dagger}(x)J_2^{(A1)}(0)$

$$T(q) = \frac{-1}{2}\frac{k_\perp^\rho}{m_b}\Gamma^\dagger\frac{\not{n}_-}{2}\gamma_{\perp\rho}\frac{\not{n}_+}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right) \tag{B.66}$$

B.5.2 One Gluon

$J_2^{(A2)\dagger}(x)J^{(A0)}(0)$

$$T(q) = g\,\frac{T^a}{2Q_-}\Gamma^\dagger\frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right)^2(2k\cdot\epsilon_\perp - l\cdot\epsilon_\perp) \tag{B.67}$$

$J_3^{(A2)\dagger}(x) J^{(A0)}(0)$

$$T(q) = g\,\frac{T^a}{4m_b}\gamma_\rho \Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right)\epsilon^{\star\rho} \tag{B.68}$$

$J^{(A0)\dagger}(x) J_3^{(A2)}(0)$

$$T(q) = g\,\frac{T^a}{4m_b}\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\gamma_\rho\left(\frac{1}{m_b-(q-k+l)_+}\right)\epsilon^{\star\rho} \tag{B.69}$$

$J_5^{(A2)\dagger}(x) J^{(A0)}(0)$

$$T(q) = g\,\frac{T^a}{2m_b}\Gamma^\dagger \frac{\not{n}_+}{2}\gamma_{\perp\rho}\frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right)\epsilon^{\star\rho} \tag{B.70}$$

$J_1^{(A1)\dagger}(x) J_2^{(A1)}(0)$

$$T(q) = g\,\frac{T^a}{2m_b}\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_{\perp\rho}\frac{\not{n}_+}{2}\Gamma\left(\frac{1}{m_b-(q-k)_+}\right)\epsilon^{\star\rho} \tag{B.71}$$

$J_1^{(A1)\dagger}(x) \mathcal{L}_\xi^{(1)}(y) J^{(A0)}(0)$

$$T(q) = g\,\frac{T^a}{m_b}\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k+l)_+}\right)\left(\frac{1}{m_b-(q-k)_+}\right)^2 l_+(k-l)\cdot\epsilon^\star \tag{B.72}$$

$J^{(A0)\dagger}(x) \mathcal{L}_{\xi 2}^{(1)}(y) J_2^{(A1)}(0)$

$$T(vq) = g\,\frac{T^a}{2m_b}\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_{\rho\perp}\Gamma\left(\frac{1}{m_b-(q-k+l)_+}\right)\left(\frac{1}{m_b-(q-k)_+}\right) l_+\epsilon^{\star\rho} \tag{B.73}$$

$J^{(A0)\dagger}(x) \mathcal{L}_{\xi 2}^{(2)}(y) J^{(A0)}(0)$

$$T(q) = g\,\frac{T^a}{2m_b}\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\left(\frac{1}{m_b-(q-k+l)_+}\right)\left(\frac{1}{m_b-(q-k)_+}\right)^2 l_+(l)\cdot\epsilon^\star \tag{B.74}$$

$J^{(A0)\dagger}(x) \mathcal{L}_{\xi 4}^{(2)}(y) J^{(A0)}(0)$

$$T(q) = g\,\frac{-T^a}{4m_b}\Gamma^\dagger \frac{\not{n}_-}{2}\gamma_\perp^\nu\gamma_\perp^\mu\Gamma\left(\frac{1}{m_b-(q-k+l)_+}\right)\left(\frac{1}{m_b-(q-k)_+}\right)(l_\nu\epsilon_\mu^\star - l_\mu\epsilon_\nu^\star) \tag{B.75}$$

B.5.3 Two Gluons

$J_2^{(A2)\dagger}(x) J(^{A0}0)$

$$T(q) = \frac{1}{2}g^2\{T^a, T^b\}\Gamma^\dagger \frac{\not{n}_-}{2}\Gamma\epsilon_\perp\cdot\epsilon_\perp\left(\frac{1}{m_b-(q-k)_+}\right)^2 \tag{B.76}$$

$J_2^{(A1)\dagger}(x)\mathcal{L}_\xi^{(1)}(y)J^{(A0)}(0)$

$$T(q) = g^2\Gamma^\dagger \frac{\not{n}_-}{2}\,\Gamma\,\frac{\epsilon_\perp\cdot\epsilon_\perp}{Q_-}\left(\frac{1}{m-(q-k)_+}\right)^2 \left(T^aT^bp_+\left(\frac{1}{m-(q-k+p)_+}\right)+T^bT^al_+\left(\frac{1}{m-(q-k+l)_+}\right)\right) \tag{B.77}$$

$J^{(A0)\dagger}(x)\mathcal{L}_\xi^{(1)}(y)\mathcal{L}_\xi^{(1)}(z)J^{(A0)}$

$$T(q) = \frac{1}{2}g^2\Gamma^\dagger \frac{\not{n}_-}{2}\,\Gamma\,\frac{2\epsilon_\perp\cdot\epsilon_\perp}{Q_-} \left(\frac{1}{m-(q-k)_+}\right)^2\left(\frac{1}{m-(q-k+l+p)_+}\right)l_+p_+ \left[\left(\frac{1}{m-(q-k+l)_+}\right)T^bT^a+T^aT^b\left(\frac{1}{m-(q-k+p)_+}\right)\right] \tag{B.78}$$

$J^{(A0)\dagger}(x)\mathcal{L}_{\xi 2}^{(2)}(y)J^{(A0)}$

$$T(q) = \frac{-1}{4}g^2\Gamma^\dagger \frac{\not{n}_-}{2}\,\Gamma\,\frac{2\epsilon_\perp\cdot\epsilon_\perp}{Q_-}[T^b,T^a] \left(\frac{1}{m-(q-k)_+}\right)^2\left(\frac{1}{m-(q-k+l+p)_+}\right)(l_+-p_+) \tag{B.79}$$

$J^{(A0)\dagger}(x)\mathcal{L}_{\xi 4}^{(2)}(y)J^{(A0)}$

$$T(q) = \frac{-1}{4}g^2\Gamma^\dagger \frac{\not{n}_-}{2}\,\gamma_\perp^\nu\gamma_{\perp\rho}\Gamma\,\frac{g_\perp^{\mu\rho}}{Q_-}[T^b,T^a] \left(\frac{1}{m-(q-k)_+}\right)\left(\frac{1}{m-(q-k+l+p)_+}\right) \left[\epsilon_\mu^*(p)\epsilon_\nu^*(l)-\epsilon_\mu^*(l)\epsilon_\nu^*(p)\right] \tag{B.80}$$

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